Quantum argument shift method for the universal enveloping algebra $U\mathfrak{gl}_d$

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July 4, 2025

Introduction

I gave a talk on the following conjecture (now a theorem) two years ago.

Theorem (I. and Sharygin, 2024)

Assumptions:

- x *and* y*: central elements of* $U\mathfrak{gl}_d$.
- ξ*: numerical matrix.*

Then $\left[\partial_\xi^m x, \partial_\xi^n y\right] = 0$ *for any* m *and* n*.*

Here

- $\partial_\xi = \operatorname{tr}(\xi\partial)$ and $\partial^i_j \in \hom U\mathfrak{gl}_d$: the quantum derivation introduced by Gurevich, Pyatov, and Saponov.

Prior Research and Motivation

Let $\mathfrak{g}$ be a Lie algebra. It is well-known that its dual $\mathfrak{g}^*$ is a Poisson manifold: $\mathfrak{g} \subset S\mathfrak{g} \subset C^\infty\mathfrak{g}^*$.

- Lie algebra $\mathfrak{g} = \mathfrak{g}^{**}$: linear functions on $\mathfrak{g}^*$.
- Symmetric algebra $S\mathfrak{g}$: polynomial functions on $\mathfrak{g}^*$.

Consider a deformation quantization of $C^\infty\mathfrak{g}^*$.

Remark

The image of the restriction of the star product on the product $S\mathfrak{g} \times S\mathfrak{g}$ is contained in the polynomial algebra $(S\mathfrak{g})[\nu]$.

Prior Research and Motivation

It makes sense to put $\nu = 1$ and obtain the star product on the symmetric algebra $S\mathfrak{g}$.

Remark

The universal enveloping algebra $U\mathfrak{g}$ is isomorphic to the symmetric algebra $S\mathfrak{g}$ with the star product.

And we have $\boxed{S\mathfrak{g} = \operatorname{gr} U\mathfrak{g}}$.

Prior Research and Motivation

Take a basis $(e_n)_{n=1}^{d}$ of $\mathfrak{g}$ and let

$$\overline{\partial}_\xi = \sum_{n=1}^{d} \xi(e_n) \frac{\partial}{\partial e_n} \in \operatorname{der} S\mathfrak{g}$$

be the directional derivative along $\forall \xi \in \mathfrak{g}^*$. Let $\overline{C}$ be the Poisson center of $S\mathfrak{g}$. The following theorem is referred to as the argument shift method.

Theorem (A. Mishchenko and A. Fomenko, 1978)

The subset $\left\{ \overline{\partial}_\xi^n x : (n, x) \in \mathbb{N} \times \overline{C} \right\}$ *is Poisson commutative.*

Prior Research and Motivation

- We obtaine a $\boxed{\text{Poisson commutative}}$ subalgebra $\overline{C}_\xi$ generated by these elements $\overline{\partial}_\xi^n x$.
- Recall $\boxed{\mathrm{gr}\, U\mathfrak{g} = S\mathfrak{g}}$.
- Vinberg asked if the argument shift algebra $\overline{C}_\xi$ can be quantised to a $\boxed{\text{commutative}}$ subalgebra C_ξ of the universal enveloping algebra $U\mathfrak{g}$ in a way that $\boxed{\mathrm{gr}\, C_\xi = \overline{C}_\xi}$.
- Such C_ξ is called a quantum argument shift algebra.

Prior Research and Motivation

- **Vinberg's problem** has been resolved in two ways:
  - Twisted Yangians: Nazarov–Olshanski.
  - Symmetrisation mapping: Tarasov.
- Also resolved using the Feigin–Frenkel center:
  - for regular elements ξ: Feigin et al. and Rybnikov.
  - for simple Lie algebras of types A and C: Futorny–Molev and Molev–Yakimova.

Motivation

The purpose of my talk is to quantize not only the algebra $\overline{C}_{\xi}$ but also the **operator** $\overline{\partial}_{\xi}$.

Quantum Derivation

Let $e = \begin{pmatrix} e^1_1 & \dots & e^1_d \\ \dots & \dots & \dots \\ e^d_1 & \dots & e^d_d \end{pmatrix}$ be a matrix satisfying the following.

- The set
$$\left\{ e^i_j : i, j = 1, \dots, d \right\}$$
is a basis of the general linear Lie algebra $\mathfrak{gl}(d, \mathbb{C})$.
- We have the commutation relations
$$\left[e^{i_1}_{j_1}, e^{i_2}_{j_2} \right] = \delta^{i_1}_{j_2} e^{i_2}_{j_1} - \delta^{i_2}_{j_1} e^{i_1}_{j_2}.$$

Quantum Derivation

We define

$$\overline{\partial} x = \begin{pmatrix} \overline{\partial}^1_1 x & \dots & \overline{\partial}^1_d x \\ \dots\dots\dots\dots\dots \\ \overline{\partial}^d_1 x & \dots & \overline{\partial}^d_d x \end{pmatrix}, \qquad \overline{\partial}^i_j = \frac{\partial}{\partial e^j_i}$$

for any element x of the symmetric algebra $S\mathfrak{gl}(d, \mathbb{C})$.

Quantum Derivation

Remark

The derivation

$$S\mathfrak{gl}_d \to M(d, S\mathfrak{gl}_d), \qquad x \mapsto \overline{\partial} x$$

is a unique linear mapping satisfying the following.

1. $\overline{\partial}\nu = 0$ for any scalar ν.
2. $\overline{\partial}\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (Leibniz rule)

$$\overline{\partial}(xy) = (\overline{\partial}x)y + x(\overline{\partial}y)$$

for any elements x and y of the symmetric algebra $S\mathfrak{gl}_d$.

There is no such mapping on $U\mathfrak{gl}_d$ because it is non-commutative.

Quantum Derivation

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quantum derivation

$$\boxed{U\mathfrak{gl}_d} \to M(d, U\mathfrak{gl}_d), \qquad x \mapsto \partial x$$

is a unique linear mapping satisfying the following.

1. $\partial \nu = 0$ for any scalar ν.
2. $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (quantum Leibniz rule)

$$\partial(xy) = (\partial x)y + x(\partial y) + \boxed{(\partial x)(\partial y)}$$

for any elements x and y of the universal enveloping algebra $U\mathfrak{gl}_d$.

Quantum Derivation

Let C be the center of the algebra $U\mathfrak{gl}_d$. Suppose that ξ is a numerical matrix and let $\partial_\xi = \mathrm{tr}(\xi\partial)$. The main theorem is the following.

Theorem (I. and Sharygin, 2024)

The subset

$$\left\{ \partial_\xi^n x : (n, x) \in \mathbb{N} \times C \right\} \tag{1}$$

is commutative.

Corollary

The subalgebra C_ξ generated by the subset (1) is the quantum argument shift algebra in the direction ξ.

Quantum Derivation

- We may assume that $\xi = \mathrm{diag}(z_1, \ldots, z_d)$ is diagonal and $(z_1, \ldots, z_d)$ is distinct considering the adjoint action of the general linear Lie group GL_d.
- Vinberg and Rybnikov showed that the quantum argument shift algebra in the direction ξ is the centralizer of the set

$$\left\{ e_i^i, \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j} \right\}_{i=1}^{d}. \tag{2}$$

- Since, by definition, the quantum argument shift algebra is <u>commutative</u>, the proof is carried out by showing that the quantum argument shift $\partial_\xi^n x$ commutes with the elements (2) by induction on the natural number n.

Formula

The center C of the algebra $U\mathfrak{gl}_d$ is the free commutative algebra on the elements

$$\operatorname{tr} e, \qquad \dots, \qquad \operatorname{tr} e^d.$$

They are called the Gelfand invariants. We would like to calculate the quantum argument shift $\partial_\xi^n x$ for a central element x. It is necessary and even sufficient to calculate the quantum derivation $\partial(e^n)^i_j$.

Formula

Remark

The linear operator

$$U\mathfrak{gl}(d,\mathbb{C}) \to M\big(d, U\mathfrak{gl}(d,\mathbb{C})\big), \quad x \mapsto \operatorname{diag}(x,\dots,x) + \partial x$$

is an algebraic homomorphism and will be denoted by ∂ from now on. We have the *quantum Leibniz rule*

$$\partial(xy) = (\partial x)(\partial y)$$

for any elements x and y of the universal enveloping algebra $U\mathfrak{gl}(d,\mathbb{C})$.

Formula

I obtained the following formula for the quantum derivation.

We define $f_\pm^{(n)}(x) = \sum_{m=0}^{n} \frac{1 \pm (-1)^{n-m}}{2} \binom{n-1}{m} x^m$.

Theorem (I, 2022)

We have

$$\begin{aligned}\partial(e^n)^i_j &= \sum_{m=0}^{n} \left(f_+^{(n-m)}(e)(e^m)^i_j + f_-^{(n-m)}(e)_j (e^m)^i\right) \\ &= \sum_{m=0}^{n} \left((e^m)^i_j f_+^{(n-m)}(e) + (e^m)_j f_-^{(n-m)}(e)^i\right).\end{aligned}$$

The formula is used for the base case.

Formula

Proof.

We assume the following form

$$\partial(e^n)^i_j = \sum_{m=0}^{n} \left(g^{(n)}_m(e)(e^m)^i_j + h^{(n)}_m(e)_j(e^m)^i\right),$$

where $g^{(n)}_m$ and $h^{(n)}_m$ are polynomials and obtain a recursive formula. □

Formula

The inductive step reduces to proving

$$\left[\operatorname{ad} e_i^i, \partial_\xi\right] = \left[\left[\operatorname{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right], \partial_\xi\right] = 0.$$

It can be done by computation.

Generators

Our theorem causes a filtration

$$C_\xi^{(0)} = C, \qquad C_\xi^{(n)} = C_\xi^{(n-1)}\left[\partial_\xi^n C\right]$$

of the quantum argument shift algebra C_ξ. Using the formula we obtain

$$C_\xi^{(1)} = C_\xi^{(0)}\Big[\operatorname{tr}\big(\xi e^n\big) : n = 1, 2, \ldots\Big],$$
$$C_\xi^{(2)} = C_\xi^{(1)}\left[\tau_\xi \begin{pmatrix} 0 & P_n^\top \\ P_m & 0 \end{pmatrix} : m, n = 0, 1, 2, \ldots\right].$$

Generators

But as for the second line these generators are redundant:

$$C_\xi^{(2)} = C_\xi^{(1)}\left[\tau_\xi \begin{pmatrix} 0 & P_n^\top \\ P_m & 0 \end{pmatrix} : \boxed{|m-n| \leq 1}\right].$$

Lemma (I, 2025)

We have

$$\sigma\begin{pmatrix} 0 & P_m^\top \\ P_{m+2n} & 0 \end{pmatrix} = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right) P_{m+k}^{(m+k)},$$

$$\sigma\begin{pmatrix} 0 & P_m^\top \\ P_{m+2n+1} & 0 \end{pmatrix} = \sum_{k=0}^{n}\binom{2n-k}{k}\left(P_{m+k+1}^{(m+k)} + P_{m+k}^{(m+k+1)}\right).$$

for any nonnegative integers m and n.

Generators

Lemma reduces to the following relations.

1. For $\varepsilon = 0, 1$,

$$\binom{x+y+n}{2n+\varepsilon} + \binom{x-y+n}{2n+\varepsilon} = \sum_{m=0}^{n} \binom{x+m}{2m+\varepsilon} \left(\binom{y+n-m}{2(n-m)} + \binom{y-1+n-m}{2(n-m)} \right).$$

2. $$\sum_{m=0}^{n} \binom{x-m}{m} \binom{y+m}{n-m} = \sum_{m=0}^{n} \binom{x+y-m}{m} \binom{m}{n-m}.$$

3. $$\binom{x}{n} = \sum_{m=0}^{n} \binom{x-m}{m} \binom{m}{n-m} + \sum_{m=0}^{n-1} \binom{x-1-m}{m} \binom{m}{n-1-m}.$$

Generators

They are shown by induction.
The generators are $\mathrm{tr}(\xi e)$, $\mathrm{tr}(\xi e^2)$, ... and

$$\mathrm{tr}(\xi^2 e),$$
$$\mathrm{tr}(2\xi^2 e^2 + \xi e \xi e),$$
$$\mathrm{tr}(\xi^2 e^3 + \xi e \xi e^2),$$
$$\mathrm{tr}(2\xi^2 e^4 + 2\xi e \xi e^3 + \xi e^2 \xi e^2 + \xi^2 e^2),$$
$$\mathrm{tr}(\xi^2 e^5 + \xi e \xi e^4 + \xi e^2 \xi e^3 + \xi^2 e^3),$$
$$\mathrm{tr}(2\xi^2 e^6 + 2\xi e \xi e^5 + 2\xi e^2 \xi e^4 + \xi e^3 \xi e^3 + 4\xi^2 e^4 + \xi e \xi e^3),$$
$$\mathrm{tr}(\xi^2 e^7 + \xi e \xi e^6 + \xi e^2 \xi e^5 + \xi e^3 \xi e^4 + 3\xi^2 e^5 + \xi e \xi e^4), \ldots.$$

They are mutually commutative.