Unbounded Operators

Yasushi Ikeda

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Contents

1 Closed Operators

Suppose that A and B are Banach spaces.

Definition 1.1
A linear mapping of a dense subspace of A into B is called a densely defined operator of A into B.

Definition 1.2
A densely defined operator is called a closed operator if the graph is closed.

Remark 1.1
The sum of a closed operator and a bounded operator is closed.

Example 1.1
Suppose that μ is a measure.

1. Suppose that x is a measurable function.

$$\{\, \xi \in L^2(\mu) : x\xi \in L^2(\mu) \,\} \to L^2(\mu), \qquad \xi \mapsto x\xi$$

   is a normal operator on $L^2(\mu)$ and the adjoint is

$$\{\, \xi \in L^2(\mu) : \overline{x}\xi \in L^2(\mu) \,\} \to L^2(\mu), \qquad \xi \mapsto \overline{x}\xi.$$

2. $M(\mu)/\{\, x : x = 0 \text{ locally a.e.} \,\}$ is a subset of the set of closed operators on $L^2(\mu)$.

Theorem 1.1 (Closed Graph Theorem)
A closed operator such that the domain is complete is bounded.

Definition 1.3
A densely defined operator extended by a closed operator is called a closable operator.

Remark 1.2
The sum of a closable operator and a bounded operator is closable.

Definition 1.4
Suppose that x is a closable operator. The unique closed operator $\overline{x}$ extending x such that

$$\operatorname{dom} \overline{x} = \min\{\, \operatorname{dom} x' : x' \text{ is a closed operator extending } x \,\}$$

is called the closure of x.

Remark 1.3
The closure of the sum of a closable operator and a bounded operator is the sum of the closure of the closable operator and the bounded operator.

Proposition 1.1
A densely defined operator is closable if and only if

$$\{\, \eta : 0 \oplus \eta \text{ is an element of the closure of the graph} \,\} = \{0\}.$$

In this case the graph of the closure is the closure of the graph.

2 Spectrum

Suppose that x is a closed operator on a complex Banach space A.

Definition 2.1
The open set

$$\rho(x) = \{\, z \in \mathbb{C} : \dim \ker(x - z) = 0 \text{ and } \operatorname{ran}(x - z) = A \,\}$$

is called the resolvent set of x.

Remark 2.1
1. Suppose that z is an element of $\rho(x)$. If $\dim A > 0$ then $B(z, \|(x - z)^{-1}\|^{-1})$ is a subset of $\rho(x)$.
2. $z \mapsto (x - z)^{-1}$ is holomorphic on $\rho(x)$ and

$$\frac{d^n}{dz^n}(x - z)^{-1} = n!((x - z)^{-1})^{n+1}$$

for $\forall n = 0, 1, \ldots$ and for $\forall z$ of $\rho(x)$.

3. Suppose that z_1 and z_2 are elements of $\rho(x)$.

$$\begin{aligned}(x-z_1)^{-1}-(x-z_2)^{-1} &= (z_1-z_2)(x-z_1)^{-1}(x-z_2)^{-1}\\ &= (z_1-z_2)(x-z_2)^{-1}(x-z_1)^{-1}.\end{aligned}$$

Example 2.1
Suppose that x is a continuous function on $\mathbb{R}^n$ (Example 1.1).

1. $\sigma(x) = \overline{x(\mathbb{R}^n)}$.
2. $\dim\ker(x-z) = 0$ if and only if $m^n(x^{-1}(\{z\})) = 0$ for $\forall z$ of $\mathbb{C}$.

3 Closable Operators on Hilbert Spaces

Suppose that H, H', H_1, H_2, H_3 are Hilbert spaces.

Remark 3.1
Suppose that x is a linear mapping of a subspace of H into H.

$$(x\xi,\eta) = \frac{1}{4}\sum_{k=0}^{3} i^k (x(\xi+i^k\eta), \xi+i^k\eta)$$

for $\forall\xi$ and $\forall\eta$ of $\operatorname{dom} x$.

Definition 3.1
Suppose that x is a closable operator of H into H'. The unique closed operator x^* of H' into H such that

$$\operatorname{dom} x^* = \{\, \eta \in H' : \xi \mapsto (x\xi,\eta) \text{ is bounded on } \operatorname{dom} x \,\}$$

and $(x\xi,\eta) = (\xi, x^*\eta)$ for $\forall\xi$ of $\operatorname{dom} x$ and for $\forall\eta$ of $\operatorname{dom} x^*$ is called the adjoint of x.

Remark 3.2
1. $\ker x^* = (\operatorname{ran} x)^\perp$.
2. The second adjoint of a closable operator is the closure.
3. The adjoint of the sum of a closable operator and a bounded operator is the sum of the adjoints.
4. If x is bounded then x^* is bounded and $\|x^*\| = \|x\|$.

Definition 3.2
Suppose that x is a conjugate linear closable operator of H into H'. The unique conjugate linear closed operator x^* of H' into H such that

$$\operatorname{dom} x^* = \{\, \eta \in H' : \xi \mapsto (\eta, x\xi) \text{ is bounded on } \operatorname{dom} x \,\}$$

and $(\eta, x\xi) = (\xi, x^*\eta)$ for $\forall\xi$ of $\operatorname{dom} x$ and for $\forall\eta$ of $\operatorname{dom} x^*$ is called the adjoint of x.

Remark 3.3 1. $\ker x^* = (\operatorname{ran} x)^\perp$.

2. The second adjoint of a conjugate linear closable operator is the closure.

3. The adjoint of the sum of a conjugate linear closable operator and a conjugate linear bounded operator is the sum of the adjoints.

4. If x is bounded then x^* is bounded and $\|x^*\| = \|x\|$.

Proposition 3.1
A densely defined operator x of H into H' is closable if

$$\{\, \eta \in H' : \xi \mapsto (x\xi, \eta) \text{ is bounded on } \operatorname{dom} x \,\}$$

is a dense subspace of H'.

Proposition 3.2
A densely defined conjugate linear operator x of H into H' is closable if

$$\{\, \eta \in H' : \xi \mapsto (\eta, x\xi) \text{ is bounded on } \operatorname{dom} x \,\}$$

is a dense subspace of H'.

Proposition 3.3
Suppose that x is a closed operator of H into H'. If x and x^* are injective then x^{-1} is a closed operator of H' into H such that $(x^{-1})^* = (x^*)^{-1}$.

Proposition 3.4
Suppose that x_1 (resp. x_2) is a closed operator of H_1 (resp. H_2) into H_2 (resp. H_3) such that x_2x_1 and $x_1^*x_2^*$ are densely defined.

1. x_2x_1 is closable and $(x_2x_1)^*$ is an extension of $x_1^*x_2^*$.

2. If x_2 is bounded then $(x_2x_1)^* = x_1^*x_2^*$.

Proposition 3.5
Suppose that x is a closed operator of H into H'. The following are equivalent.

1. x is a partial isometry.

2. x^*x is a projection.

3. x^* is a partial isometry.

4. xx^* is a projection.

In this case x^*x is a projection onto $(\ker x)^\perp$ and xx^* is a projection onto $\operatorname{ran} x$.

Proposition 3.6
Suppose that x is a conjugate linear closed operator of H into H'. The following are equivalent.

1. x is a conjugate linear partial isometry.

2. x^*x is a projection.

3. x^* is a conjugate linear partial isometry.

4. xx^* is a projection.

In this case x^*x is a projection onto $(\ker x)^\perp$ and xx^* is a projection onto $\operatorname{ran} x$.

Proposition 3.7
Suppose that x is a closed operator of H into H'.

$$H \oplus H' = \{\, \xi \oplus (-x\xi) : \xi \in \operatorname{dom} x \,\} \oplus \{\, (x^*\eta) \oplus \eta : \eta \in \operatorname{dom} x^* \,\}$$

and

$$\|(xx^* + 1)^{-1}x\| \leq 1, \qquad \overline{(xx^* + 1)^{-1}x} = x(x^*x + 1)^{-1}.$$

Definition 3.3
A dense subspace of the domain of a closed operator is called a core for the operator if the operator is the closure of the restriction of the operator to the subspace.

Theorem 3.1 (Polar Decomposition)
Suppose that x is a closed operator of H into H'.

1. There exists a unique partial isometry $p(x)$ of H into H' such that $\ker p(x) = \ker(x^*x)^{1/2}$ and $x = p(x)(x^*x)^{1/2} = (xx^*)^{1/2}p(x)$.

2. $p(x)f(x^*x) = f(xx^*)p(x)$ for a measurable function $\forall f$ on $(0, \infty)$.

3. $p(x^*) = p(x)^*$.

Remark 3.4 1. If $\ker p = \ker r$ then $x = pr$ is a closed operator of H into H' such that $p(x) = p$ and $(x^*x)^{1/2} = r$ for a partial isometry $\forall p$ of H into H' and for a non-negative self-adjoint operator $\forall r$ on H.

2. If $\ker p^* = \ker r$ then $x = rp$ is a closed operator of H into H' such that $p(x) = p$ and $(xx^*)^{1/2} = r$ for a partial isometry $\forall p$ of H into H' and for a non-negative self-adjoint operator $\forall r$ on H'.

Theorem 3.2 (Polar Decomposition)
Suppose that x is a conjugate linear closed operator of H into H'.

1. There exists a unique conjugate linear partial isometry $p(x)$ of H into H' such that $\ker p(x) = \ker(x^*x)^{1/2}$ and $x = p(x)(x^*x)^{1/2} = (xx^*)^{1/2}p(x)$.

2. $p(x)f(x^*x) = f(xx^*)^*p(x)$ for a measurable function $\forall f$ on $(0, \infty)$.

3. $p(x^*) = p(x)^*$.

Remark 3.5 1. If $\ker p = \ker r$ then $x = pr$ is a conjugate linear closed operator of H into H' such that $p(x) = p$ and $(x^*x)^{1/2} = r$ for a conjugate linear partial isometry $\forall p$ of H into H' and for a non-negative self-adjoint operator $\forall r$ on H.

2. If $\ker p^* = \ker r$ then $x = rp$ is a conjugate linear closed operator of H into H' such that $p(x) = p$ and $(xx^*)^{1/2} = r$ for a conjugate linear partial isometry $\forall p$ of H into H' and for a non-negative self-adjoint operator $\forall r$ on H'.

Definition 3.4
A densely defined operator is called a symmetric operator if the numerical range is real.

Remark 3.6

1. Suppose that x is a symmetric operator.

$$\|(x - z)\xi\| \geq |\mathrm{Im}\, z| \|\xi\|$$

for $\forall z$ of $\mathbb{C}$ and for $\forall \xi$ of $\mathrm{dom}\, x$.

2. Suppose that x is a closed symmetric operator.

$$|\mathrm{Im}\, z| \|(x - z)^{-1}\| \leq 1$$

for $\forall z$ of $\rho(x)$.

3. $\mathrm{ran}(x - z)$ is closed for a closed symmetric operator $\forall x$ and for $\forall z$ of $\mathbb{C} \setminus \mathbb{R}$.

4. The spectrum of a self-adjoint operator is real.

5. A symmetric operator is closable and extended by the adjoint.

6. The closure of a symmetric operator is a symmetric operator.

Proposition 3.8
Suppose that x is a symmetric operator on H. The following are equivalent.

1. x is self-adjoint.

2. x is closed and $\dim \ker(x - i)^* = \dim \ker(x + i)^* = 0$.

3. $\mathrm{ran}(x - i) = \mathrm{ran}(x + i) = H$.

Theorem 3.3 (Stone's Theorem)
The set of strongly continuous (resp. norm continuous) one parameter unitary groups is the set of self-adjoint (resp. bounded self-adjoint) operators.

Remark 3.7
Suppose that x is a self-adjoint operator.

$$\lim_{k \to 0} \frac{e^{ikx} - 1}{ik} \xi = \left[\frac{1}{i} \frac{d}{dk} (e^{ikx} \xi) \right]_{k=0}$$

exists if and only if ξ is an element of $\mathrm{dom}\, x$ for $\forall \xi$.

$$\frac{1}{i} \frac{d}{dk} (e^{ikx} \xi) = x e^{ikx} \xi$$

for $\forall \xi$ of $\mathrm{dom}\, x$.

Definition 3.5
A symmetric operator is said to be essentially self-adjoint if the closure is self-adjoint.

Proposition 3.9
Suppose that x is a symmetric operator on H. The following are equivalent.

1. x is essentially self-adjoint.
2. $\dim\ker(x-i)^* = \dim\ker(x+i)^* = 0$.
3. $\overline{\mathrm{ran}}\,(x-i) = \overline{\mathrm{ran}}\,(x+i) = H$.

Definition 3.6
A symmetric operator is said to be non-negative if the numerical range is non-negative.

Remark 3.8
1. Suppose that x is a non-negative symmetric operator.
$$\|(x+x_0)\xi\| \geq x_0\|\xi\|$$
for $\forall x_0 \geq 0$ and for $\forall \xi$ of $\operatorname{dom} x$.
2. $\operatorname{ran}(x+x_0)$ is closed for a closed non-negative symmetric operator $\forall x$ and for $\forall x_0 > 0$.
3. The spectrum of a non-negative self-adjoint operator is non-negative.

Proposition 3.10
If x is a closed operator of H into H' then x^*x is a non-negative self-adjoint operator on H and $\operatorname{dom} x^*x$ is a core for x.

Remark 3.9
1. Suppose that x is a closed operator of H into H'.
$$\operatorname{dom} x \cap (\ker x)^{\perp} \to \overline{\mathrm{ran}}\, x, \qquad\qquad \xi \mapsto x\xi$$
is a closed operator of $(\ker x)^{\perp}$ into $\overline{\mathrm{ran}}\, x$.
2. Suppose that x is a (resp. non-negative) self-adjoint operator.
$$\operatorname{dom} x \cap (\ker x)^{\perp} \to (\ker x)^{\perp}, \qquad\qquad \xi \mapsto x\xi$$
is a (resp. non-negative) self-adjoint operator on $(\ker x)^{\perp}$.

Remark 3.10
Suppose that $H = \bigoplus_i H_i$ and $H' = \bigoplus_i H'_i$ are direct sum Hilbert spaces.

1. If $(x_i)_i$ is a family such that x_i is a closed operator of H_i into H'_i for $\forall i$ then
$$x = \bigoplus_i x_i \colon \left\{ \xi \in \prod_i \operatorname{dom} x_i : \sum_i \|\xi_i\|^2, \sum_i \|x_i\xi_i\|^2 < \infty \right\} \to H',$$
$$\xi \mapsto \bigoplus_i x_i\xi_i$$
is a closed operator of H into H' such that $x^* = \bigoplus_i x_i^*$ and $x^*x = \bigoplus_i x_i^* x_i$.

2. If $(x_i)_i$ is a family such that x_i is a normal operator on H_i for $\forall i$ then $x = \bigoplus_i x_i$ is a normal operator on H.

3. If $(x_i)_i$ is a family such that x_i is a (resp. non-negative) self-adjoint operator on H_i for $\forall i$ then $x = \bigoplus_i x_i$ is a (resp. non-negative) self-adjoint operator on H.

4 Spectral Measures

Suppose that X is a set.

Definition 4.1

A mapping P of a σ-algebra over X into the set of projections of a von Neumann algebra is called a spectral measure if P is countably additive with respect to the ultratopologies and $P(X) = 1$.

Remark 4.1

Suppose that P is a spectral measure.

1. $P(S_1 \cap S_2) = P(S_1)P(S_2)$ for $\forall S_1$ and $\forall S_2$.

2. $S \mapsto \varphi(P(S))$ is a probability measure for a normal state $\forall \varphi$.

Proposition 4.1

A mapping P of a σ-algebra into the set of projections of a von Neumann algebra such that $S \mapsto (P(S)\xi, \xi)$ is a probability measure for a unit vector $\forall \xi$ is a spectral measure.

Definition 4.2

Suppose that P is a spectral measure. We denote the *-algebra of equivalence classes of elements of the *-algebra of measurable functions by $M(P)$.

Definition 4.3

A closed operator is said to be affiliated with a von Neumann algebra if it commutes with any unitary element of the commutant.

Proposition 4.2

A normal operator is affiliated with a von Neumann algebra if and only if the resolution of the identity is affiliated with the von Neumann algabra.

Theorem 4.1

Suppose that x' is a closed operator on H and that S is a self-adjoint subset of $B(H)$. The following are equivalent.

1. x' is affiliated with S'.

2. $x\xi$ is an element of $\operatorname{dom} x'$ and $x'x\xi = xx'\xi$ for $\forall x$ of S and for $\forall \xi$ of $\operatorname{dom} x'$.

Proposition 4.3

Suppose that P is a spectral measure affiliated with a von Neumann algebra N.

$$L^\infty(P) = \left\{ f \in M(P) : \|f\|_\infty = \min\{\, m : |f| \le m \,\} \right.$$
$$\left. = \min_{P(N)=0} \sup_{x \in X \setminus N} |f(x)| < \infty \right\}$$

is a C^*-subalgebra of N.

Remark 4.2

Suppose that N_* is separable.

1. There exists a finite measure mutually absolutely continuous with respect to P. We denote the Banach space of complex measures absolutely continuous with respect to P by $L^1(P)$.
$$L^\infty(P) = L^1(P)^*.$$
2. $L^\infty(P)$ is a von Neumann subalgebra of N and $L^\infty(P)_* = L^1(P)$.

Theorem 4.2

Suppose that P is a spectral measure.

1. Suppose that f is a measurable function. There exists a unique closed operator
$$\int f(x)P(dx) = \int xP(f^{-1}(dx))$$
such that
$$\mathrm{dom} \int f(x)P(dx) = \left\{ \xi : \int |f(x)|^2 P_\xi(dx) < \infty \right\}$$
and
$$(\int f(x)P(dx)\xi, \eta) = \int f(x)P_{\xi,\eta}(dx)$$
for $\forall \xi$ of $\mathrm{dom} \int f(x)P(dx)$ and for $\forall \eta$.
$$(\int f(x)P(dx))^* = \int \overline{f(x)}P(dx)$$
and
$$\int \overline{f(x)}P(dx) \int f(x)P(dx) = \int |f(x)|^2 P(dx).$$
If f is invertible then
$$\int f(x)P(dx)$$
is injective and
$$(\int f(x)P(dx))^{-1} = \int f(x)^{-1}P(dx).$$

2. Suppose that f is a real measurable function.

$$\int f(x)P(dx)$$

is self-adjoint. If f is non-negative then

$$\int f(x)P(dx)$$

is non-negative and

$$(\int f(x)P(dx))^{1/2} = \int f(x)^{1/2}P(dx).$$

3. Suppose that f is a measurable function.

$$\int f(x)P(dx)P(S)$$

is an extension of

$$P(S)\int f(x)P(dx)$$

for $\forall S$.

4. Suppose that f and g are measurable functions.

$$\int |f(x)g(x)||P_{\xi,\eta}|(dx) \leq (\int |f(x)|^2 P_\xi(dx))^{1/2}(\int |g(x)|^2 P_\eta(dx))^{1/2}$$

for $\forall\xi$ and $\forall\eta$.

$$(\int f(x)P(dx)\xi, \int g(x)P(dx)\eta) = \int f(x)\overline{g(x)}P_{\xi,\eta}(dx)$$

for $\forall\xi$ of dom $\int f(x)P(dx)$ and for $\forall\eta$ of dom $\int g(x)P(dx)$.

5. Suppose that f and g are measurable functions.

$$\begin{aligned}\operatorname{dom}&\Big(\int f(x)P(dx) + \int g(x)P(dx)\Big)\\ &= \operatorname{dom}\int f(x)P(dx) \cap \operatorname{dom}\int (f+g)(x)P(dx)\\ &= \operatorname{dom}\int g(x)P(dx) \cap \operatorname{dom}\int (f+g)(x)P(dx)\end{aligned}$$

and

$$\int (f+g)(x)P(dx)$$

is an extension of

$$\int f(x)P(dx) + \int g(x)P(dx).$$

6. Suppose that f and g are measurable functions.

$$\operatorname{dom} \int g(x)P(dx) \int f(x)P(dx) = \operatorname{dom} \int f(x)P(dx) \cap \operatorname{dom} \int f(x)g(x)P(dx)$$

   and

$$\int f(x)g(x)P(dx)$$

   is an extension of

$$\int g(x)P(dx) \int f(x)P(dx).$$

7. Suppose that f is a measurable function.

$$\| \int f(x)P(dx)\| = \|f\|_\infty.$$

Definition 4.4
A closed operator is called a normal operator if it commutes with the adjoint.

Theorem 4.3 (Spectral Theorem)
There exists a unique spectral measure P_x on the complex plane such that

$$x = \int x' P_x(dx')$$

for a normal operator $\forall x$.

Remark 4.3
The support of the resolution of the identity is the spectrum.

Proposition 4.4

$$\|x\| = \sup_{x' \in \sigma(x)} |x'|$$

for a normal operator $\forall x$.

Theorem 4.4 (Spectral Mapping Theorem)
Suppose that x is a normal operator.

$$\sigma(f(x)) = \overline{f(\sigma(x))}$$

for a continuous function $\forall f$ on $\sigma(x)$.

Proposition 4.5
$x^{1/2}$ is the unique non-negative self-adjoint operator such that $x^{1/2}x^{1/2} = x$ for a non-negative self-adjoint operator $\forall x$.

Theorem 4.5

$$\frac{1}{x-z} = i\int_0^\infty e^{-ik(x-z)}\,dk$$

for a self-adjoint operator $\forall x$ and for $\forall z$ of the upper half-plane.

Proposition 4.6

$$\ker(x - x_0) = \operatorname{ran} P_x(\{x_0\})$$

for a self-adjoint operator $\forall x$ and for a real number $\forall x_0$.

Proposition 4.7
Suppose that $(P_i)_i$ is a family of spectral measures on a measurable space.

$$S \mapsto (\bigoplus_i P_i)(S) = \bigoplus_i P_i(S)$$

is a spectral measure and

$$\int f(x)(\bigoplus_i P_i)(dx) = \bigoplus_i \int f(x)P_i(dx)$$

for a measurable function $\forall f$.

Proposition 4.8
If $x = \bigoplus_i x_i$ is a direct sum normal operator then $P_x = \bigoplus_i P_{x_i}$ and $f(x) = \bigoplus_i f(x_i)$ for a measurable function $\forall f$ on a Borel set such that $P_x(\operatorname{dom} f) = 1$.

Remark 4.4
If $x = \bigoplus_i x_i$ is a direct sum normal operator then $\sigma(x) = \overline{\bigcup_i \sigma(x_i)}$.

Proposition 4.9

1. Suppose that P is a spectral measure and that f is a measurable function on a measurable set such that $P(\operatorname{dom} f) = 1$. If g is a measurable function on a Borel set such that $P(f^{-1}(\operatorname{dom} g)) = 1$ then

$$g\left(\int_{\operatorname{dom} f} f(x)P(dx)\right) = \int_{f^{-1}(\operatorname{dom} g)} (g\circ f)(x)P(dx).$$

2. Suppose that x is a normal operator and that f is a measurable function on a Borel set such that $P_x(\operatorname{dom} f) = 1$. If g is a measurable function on a Borel set such that $P_{f(x)}(\operatorname{dom} g) = P_x(f^{-1}(\operatorname{dom} g)) = 1$ then $g(f(x)) = (g\circ f)(x)$.

Remark 4.5
Suppose that J is a conjugate linear orthogonal operator of H onto H'.

1. If x is a closable operator on H then $(JxJ^*)^* = Jx^*J^*$.

2. If P is a spectral measure relative to H then

$$S \mapsto JP(S)J^*$$

is a spectral measure relative to H' and

$$J(\int f(x)P(dx))J^* = \int \overline{f(x)}JP(dx)J^*$$

for a measurable function $\forall f$.

3. If x is a self-adjoint operator on H then JxJ^* is a self-adjoint operator on H' and

$$P_{JxJ^*}(S) = JP_x(S)J^*$$

for a Borel set $\forall S$.

5 Generalized Spectral Measures

Definition 5.1

A mapping Π of a σ-algebra over X into the set of positive elements of a von Neumann algebra is called a generalized spectral measure if Π is countably additive with respect to the ultratopologies and $\Pi(X) = 1$.

Proposition 5.1

A mapping Π of a σ-algebra into the set of positive elements of a von Neumann algebra such that $S \mapsto (\Pi(S)\xi, \xi)$ is a probability measure for a unit vector $\forall \xi$ is a generalized spectral measure.

References

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