Tomita's Theorem

Yasushi Ikeda

September 8, 2022

Contents

1 Tomita's Theorem - Shortest Course

1.1 Closed operators associated with the cyclic and separating vector

Suppose that N is a von Neumann algebra on a Hilbert space H and ξ is a cyclic and separating vector for N.

We define a conjugate linear operator S_0 on $N\xi$ by

$$S_0 x\xi = x^*\xi.$$

Then we have $S_0 = S_0^{-1}$. Similarly, we define an operator F_0 on $N'\xi$ by

$$F_0 x'\xi = x'^*\xi.$$

Suppose that (x, x') is any element of $N \times N'$. Then we have

$$(x'\xi, S_0 x\xi) = (x'\xi, x^*\xi) = (x\xi, x'^*\xi) = (x\xi, F_0 x'\xi).$$

Therefore, the operators S_0 and F_0 are closable. We define $S = \overline{S_0}$ and $F = \overline{F_0}$. Then the operator F is extended by S^*.

Lemma 1.1
Suppose that $S_0 = S_0^{-1}$ is a conjugate linear closable operator on a Hilbert space H and let $S = \overline{S_0}$. Then we have $S = S^{-1}$ and $S^* = S^{*-1}$.

Proof. It is sufficient to show that $S^* = S^{*-1}$ because if we show $S_0^* = S^* = S^{*-1} = S_0^{*-1}$, then we have $S = S^{**} = S^{**-1} = S^{-1}$. Suppose that ξ is an element of $\operatorname{dom} S_0$ and η is an element of $\operatorname{dom} S^*$. Then we have $(S^*\eta, S_0\xi) = (\xi, \eta)$. Therefore, the vector $S^*\eta$ belongs to $\operatorname{dom} S^*$ and $S^*S^*\eta = \eta$. □

Therefore, we have $S = S^{-1}$ and $S^* = S^{*-1}$.

PROPOSITION 1.1
Suppose that η is an element of $\operatorname{dom} S^*$. Then the operator

$$x\xi \mapsto x\eta, \qquad N\xi \to H$$

is closable. We write x'_η for its closure. Then the operator x'_η is affiliated with N' and the operator $x'_{S^*\eta}$ is extended by x'^*_η.

Proof. We define an operator a_0 of $N\xi$ into H by $a_0 x\xi = x\eta$. Similarly, we define an operator b_0 of $N\xi$ into H by $b_0 x\xi = xS^*\eta$. Then for each x, $y \in N$, we have

$$\begin{aligned}(a_0 x\xi, y\xi) &= (\eta, x^* y\xi) \\ &= (\eta, Sy^* x\xi) \\ &= (y^* x\xi, S^*\eta) \\ &= (x\xi, yS^*\eta) \\ &= (x\xi, b_0 y\xi).\end{aligned}$$

Therefore, the operators a_0 and b_0 are closable and the operator $x'_{S^*\eta} = \overline{b_0}$ is extended by $x'^*_\eta = a_0^*$. Suppose that x is an element of N and $\overline{\xi}$ is an element of $\operatorname{dom} x'_\eta$. Then there exists a sequence $(x_n)_{n=1}^\infty$ of N such that

$$(\overline{\xi}, x'_\eta \overline{\xi}) = \lim_{n\to\infty} (x_n\xi, x_n\eta).$$

Then we have

$$x x'_\eta \overline{\xi} = \lim_{n\to\infty} x x_n \eta = \lim_{n\to\infty} x'_\eta x x_n \xi = x'_\eta x \overline{\xi}.$$

Therefore, the operator x'_η is affiliated with N'. □

Suppose that η is an element of $\operatorname{dom} S^*$ and let $x' = x'_\eta$. Suppose that $x' = v'|x'|$ is a polar decomposition and

$$|x'| = \int_0^\infty s P'(ds)$$

is a spectral decomposition. We define $P'_n = P'([0, n])$ and

$$x'_n = x' P'_n = v' \int_0^n s P'(ds) \in N'.$$

Then we have

$$\begin{aligned}\|\eta - x'_n\xi\|^2 &= \|x'\xi - x'P'_n\xi\|^2 \\ &\le \|(|x'| - |x'|P'_n)\xi\|^2 \\ &= \int_{(n,\infty)} s^2\|P'(ds)\xi\|^2 \to 0\end{aligned}$$

and

$$\begin{aligned}\|S^*\eta - x'^*_n\xi\|^2 &= \||x'|v'^*\xi - |x'|P'_n v'^*\xi\|^2 \\ &= \int_{(n,\infty)} s^2\|P'(ds)v'^*\xi\|^2 \to 0.\end{aligned}$$

Therefore, the vector η belongs to $\operatorname{dom} F$ and we have $F = S^*$. We define

$$C_C(0,\infty)_+ = \big\{\, f\colon [0,\infty) \to [0,\infty) : f \text{ is continuous} \\ \text{and } \operatorname{supp} f \text{ is contained in } (0,\infty)\,\big\}.$$

Proposition 1.2

Suppose that η is an element of $\operatorname{dom} S^*$. Then the vector $f(x'_\eta x'^*_\eta)\eta$ belongs to $\operatorname{dom} S^*$ and we have $S^* f(x'_\eta x'^*_\eta)\eta = f(x'^*_\eta x'_\eta)S^*\eta$ for each element f of $C_C(0,\infty)_+$.

Proof. There exists an element g of $C_C(0,\infty)_+$ such that $f(x) = xg(x)$ for $x > 0$. Suppose that x is an element of N. Then we have

$$\begin{aligned}\big(f(x'_\eta x'^*_\eta)\eta, S_0 x\xi\big) &= \big(x'^*_\eta g(x'_\eta x'^*_\eta)\eta, x^*S^*\eta\big) \\ &= \big(x'^*_\eta g(x'_\eta x'^*_\eta)x\eta, S^*\eta\big) \\ &= \big(x'^*_\eta g(x'_\eta x'^*_\eta)x'_\eta x\xi, S^*\eta\big) \\ &= \big(f(x'^*_\eta x'_\eta)x\xi, S^*\eta\big) \\ &= \big(x\xi, f(x'^*_\eta x'_\eta)S^*\eta\big).\end{aligned}$$

Therefore, the vector $f(x'_\eta x'^*_\eta)\eta$ belongs to $\operatorname{dom} S^*$ and we have $S^* f(x'_\eta x'^*_\eta)\eta = f(x'^*_\eta x'_\eta)S^*\eta$. □

Proposition 1.3

$$N' = \{\, x'_\eta : \eta \text{ is an element of } \operatorname{dom} S^* \text{ such that } x'_\eta \text{ is bounded} \,\}.$$

Proof. Suppose that x' is an element of N'. We define an element $\eta = x'\xi$ of $\operatorname{dom} S^*$. Suppose that x is an element of N. Then we have $x'_\eta x\xi = xx'\xi = x'x\xi$. Therefore, the operator x'_η is bounded and $x' = x'_\eta$. □

1.2 Polar decomposition - interlude

Suppose that $S = S^{-1}$ is a conjugate linear closed operator on a Hilbert space H and let $S = J|S|$ be the polar decomposition.

Then $\ker J = \ker S = \{0\}$ and the operator J is a conjugate linear isometry. Since the operator $|S|^2$ is injective and positive, we can define a self-adjoint operator $h = -\log|S|^2$. Then we have

$$S = J\exp\left(-\frac{h}{2}\right).$$

REMARK 1.1
There exists a unique pair (J, h) of a conjugate linear isometry J and a (possibly unbounded) self-adjoint operator h such that

$$S = J\exp\left(-\frac{h}{2}\right).$$

Since $\ker J^* = \ker S^* = \{0\}$, we have $J^* = J^{-1}$. Since $|S^*|^2 = SS^* = (S^*S)^{-1} = |S|^{-2}$, we have

$$|S^*| = \exp\left(\frac{h}{2}\right).$$

Therefore, we have

$$J^*\exp\left(\frac{h}{2}\right) = S^* = S^{*-1} = J\exp\left(\frac{h}{2}\right)$$

and we have $J = J^*$.

REMARK 1.2
The relation $J = J^* = J^{-1}$ holds and

$$S = J\exp\left(-\frac{h}{2}\right) = \exp\left(\frac{h}{2}\right)J, \qquad S^* = J\exp\left(\frac{h}{2}\right) = \exp\left(-\frac{h}{2}\right)J.$$

1.3 Key propositions

LEMMA 1.2
Suppose that μ is a Borel measure on $[0,\infty)$ and $x_0 \geq 0$ such that

$$\int_0^\infty x^2 f(x)\mu(dx) \leq x_0^2 \int_0^\infty f(x)\mu(dx)$$

for each element f of $C_C(0,\infty)_+$. Then we have $\mu((x_0,\infty)) = 0$.

Proof. Suppose that $[a, b]$ is any subinterval of (x_0,∞). Then there exists an element f of $C_C(0,\infty)_+$ such that

$$f(x) = \frac{1}{x^2 - x_0^2}$$

for $a \leq x \leq b$. Then we have

$$\mu([a,b]) = \int_a^b (x^2 - x_0^2) f(x)\mu(dx) \leq 0.$$

Therefore, we have $\mu((x_0,\infty)) = 0$. □

PROPOSITION 1.4
Suppose that x is an element of N and ω is a real number. Then there exists a unique element x'_ω of N' such that

$$J\left(\cosh \frac{h+\omega}{2}\right)^{-1} x\xi = x'_\omega \xi$$

and we have $\|x'_\omega\| \leq \|x\|$.

Proof. We define

$$\eta = J\left(\cosh \frac{h+\omega}{2}\right)^{-1} x\xi \in \operatorname{dom} S \cap \operatorname{dom} S^*$$

and let $x' = x'_\eta$. Suppose that $x' = v'|x'|$ is a polar decomposition and

$$|x'^*| = \int_0^\infty sP'(ds)$$

is a spectral decomposition. By Proposition 1.3, it is sufficient to show that $\|x'\| \leq \|x\|$. Suppose that f is any element of $C_C(0,\infty)_+$. Since

$$x\xi = \left(\cosh \frac{h+\omega}{2}\right) J\eta = \frac{1}{2}\exp\left(\frac{\omega}{2}\right) S\eta + \frac{1}{2}\exp\left(-\frac{\omega}{2}\right) S^*\eta,$$

we have

$$\begin{aligned}
&\||x'|\sqrt{f}(|x'|)x\xi\|^2 \\
&\quad \geq 4\operatorname{Re}\left(|x'|\sqrt{f}(|x'|)\frac{1}{2}\exp\left(\frac{\omega}{2}\right)S\eta, |x'|\sqrt{f}(|x'|)\frac{1}{2}\exp\left(-\frac{\omega}{2}\right)S^*\eta\right) \\
&\qquad = \operatorname{Re}\big(S\eta, |x'|^2 f(|x'|)S^*\eta\big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\||x'|\sqrt{f}(|x'|)x\xi\|^2 &\geq \operatorname{Re}\big(S\eta, |x'|^2 f(|x'|)S^*\eta\big) \\
&= \operatorname{Re}(|x'^*|^2 f(|x'^*|)\eta, \eta) \\
&= \||x'^*|\sqrt{f}(|x'^*|)\eta\|^2,
\end{aligned}$$

$$
\begin{aligned}
\||x'^*|\sqrt{f}(|x'^*|)\eta\|^2 &\le \||x'|\sqrt{f}(|x'|)x\xi\|^2 \\
&\le \|x\|^2\||x'|\sqrt{f}(|x'|)\xi\|^2 \\
&= \|x\|^2\|\sqrt{f}(|x'|)|x'|\xi\|^2 \\
&= \|x\|^2\|\sqrt{f}(|x'|)v'^*\eta\|^2 \\
&\le \|x\|^2\|\sqrt{f}(|x'^*|)\eta\|^2,
\end{aligned}
$$

$$
\int_0^\infty s^2 f(s)\|P'(ds)\eta\|^2 \le \|x\|^2 \int_0^\infty f(s)\|P'(ds)\eta\|^2.
$$

By Lemma 1.2, we have $P'((\|x\|, \infty))\eta = 0$. We define $P' = P'([0, \|x\|])$. Then we have $\eta = P'\eta$. Suppose that y is any element of N. Then we have

$$
\begin{aligned}
x'y\xi &= y\eta \\
&= P'y\eta \\
&= P'x'y\xi \\
&= \int_0^{\|x\|} sP'(ds)v'y\xi.
\end{aligned}
$$

Therefore, we have $\|x'\| \le \|x\|$. □

Proposition 1.5

Suppose that ζ_1 and ζ_2 are elements of

$$
\operatorname{dom}\exp\left(\frac{h}{2}\right) \cap \operatorname{dom}\exp\left(-\frac{h}{2}\right).
$$

Then we have

$$
\begin{aligned}
(JxJ\zeta_1, \zeta_2) = \frac{1}{2}\exp\left(\frac{\omega}{2}\right)&\left(x'_\omega \exp\left(\frac{h}{2}\right)\zeta_1, \exp\left(-\frac{h}{2}\right)\zeta_2\right) \\
&+ \frac{1}{2}\exp\left(-\frac{\omega}{2}\right)\left(x'_\omega \exp\left(-\frac{h}{2}\right)\zeta_1, \exp\left(\frac{h}{2}\right)\zeta_2\right).
\end{aligned}
$$

Proof. Suppose that a and b are any elements of N. By Proposition 1.4, there exist elements a' and b' of N' such that

$$
J\left(\cosh\frac{h}{2}\right)^{-1} a\xi = a'\xi, \qquad J\left(\cosh\frac{h}{2}\right)^{-1} b\xi = b'\xi.
$$

We have

$$
x\xi = \frac{1}{2}\exp\left(\frac{\omega}{2}\right)Sx'_\omega\xi + \frac{1}{2}\exp\left(-\frac{\omega}{2}\right)S^*x'_\omega\xi.
$$

Since

$$\begin{aligned}(a'^*b'\xi, x\xi) &= (b'\xi, xa'\xi)\\ &= \left(J\left(\cosh\frac{h}{2}\right)^{-1}b\xi, xJ\left(\cosh\frac{h}{2}\right)^{-1}a\xi\right)\\ &= \left(\left(\cosh\frac{h}{2}\right)^{-1}JxJ\left(\cosh\frac{h}{2}\right)^{-1}a\xi, b\xi\right),\end{aligned}$$

$$\begin{aligned}(a'^*b'\xi, Sx'_\omega\xi) &= (x'_\omega\xi, b'^*a'\xi)\\ &= (S^*x'^*_\omega S^*b'\xi, a'\xi)\\ &= \left(S^*x'^*_\omega S^*J\left(\cosh\frac{h}{2}\right)^{-1}b\xi, J\left(\cosh\frac{h}{2}\right)^{-1}a\xi\right)\\ &= \left(\exp\left(-\frac{h}{2}\right)\left(\cosh\frac{h}{2}\right)^{-1}x'_\omega\exp\left(\frac{h}{2}\right)\left(\cosh\frac{h}{2}\right)^{-1}a\xi, b\xi\right),\end{aligned}$$

$$\begin{aligned}(a'^*b'\xi, S^*x'_\omega\xi) &= (b'\xi, a'S^*x'_\omega\xi)\\ &= (b'\xi, S^*x'_\omega S^*a'\xi)\\ &= \left(J\left(\cosh\frac{h}{2}\right)^{-1}b\xi, S^*x'_\omega S^*J\left(\cosh\frac{h}{2}\right)^{-1}a\xi\right)\\ &= \left(\exp\left(\frac{h}{2}\right)\left(\cosh\frac{h}{2}\right)^{-1}x'_\omega\exp\left(-\frac{h}{2}\right)\left(\cosh\frac{h}{2}\right)^{-1}a\xi, b\xi\right),\end{aligned}$$

we have

$$\begin{aligned}&\left(\cosh\frac{h}{2}\right)^{-1}JxJ\left(\cosh\frac{h}{2}\right)^{-1}\\ &\quad= \frac{1}{2}\exp\left(\frac{\omega}{2}\right)\exp\left(-\frac{h}{2}\right)\left(\cosh\frac{h}{2}\right)^{-1}x'_\omega\exp\left(\frac{h}{2}\right)\left(\cosh\frac{h}{2}\right)^{-1}\\ &\qquad+\frac{1}{2}\exp\left(-\frac{\omega}{2}\right)\exp\left(\frac{h}{2}\right)\left(\cosh\frac{h}{2}\right)^{-1}x'_\omega\exp\left(-\frac{h}{2}\right)\left(\cosh\frac{h}{2}\right)^{-1}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(JxJ\zeta_1, \zeta_2) &= \frac{1}{2}\exp\left(\frac{\omega}{2}\right)\left(x'_\omega\exp\left(\frac{h}{2}\right)\zeta_1, \exp\left(-\frac{h}{2}\right)\zeta_2\right)\\ &\quad+\frac{1}{2}\exp\left(-\frac{\omega}{2}\right)\left(x'_\omega\exp\left(-\frac{h}{2}\right)\zeta_1, \exp\left(\frac{h}{2}\right)\zeta_2\right). \quad\square\end{aligned}$$

1.4 Complex analysis - interlude

THEOREM 1.1
Suppose that f is a bounded continuous function of

$$\left\{ z \in \mathbb{C} : |\operatorname{Im} z| \le \frac{1}{2} \right\}$$

into the complex plane which is holomorphic on

$$\left\{ z \in \mathbb{C} : |\operatorname{Im} z| < \frac{1}{2} \right\}.$$

Then we have

$$f(0) = \int_{-\infty}^{\infty} e^{\theta t} \left(\frac{1}{2} \exp\left(-\frac{i\theta}{2} \right) f\left(t - \frac{i}{2} \right) + \frac{1}{2} \exp\left(\frac{i\theta}{2} \right) f\left(t + \frac{i}{2} \right) \right) \frac{dt}{\cosh(\pi t)}$$

for each $-\pi < \theta < \pi$.

Proof. We define $U = \left\{ z \in \mathbb{C} : |\operatorname{Im} z| < \dfrac{1}{2} \right\}$ and

$$\varphi(z) = \frac{\pi e^{\theta z} f(z)}{\sinh(\pi z)}.$$

Then the function φ is holomorphic on $\{ z \in \mathbb{C} : |\operatorname{Im} z| < 1 \} \setminus \{0\}$ and we have

$$f(0) = \lim_{z \to 0} z\varphi(z) = \frac{1}{2\pi i} \left(\int_{-r}^{r} \left(\varphi\left(t - \frac{i}{2} \right) - \varphi\left(t + \frac{i}{2} \right) \right) dt \right.$$
$$\left. + i \int_{-2^{-1}}^{2^{-1}} \big(\varphi(r + it) - \varphi(-r + it) \big)\, dt \right)$$

for each $r > 0$. We define

$$\|f\| = \sup_{|\operatorname{Im} z| \le 2^{-1}} |f(z)|.$$

Then we have

$$\left| \int_{-2^{-1}}^{2^{-1}} \big(\varphi(r + it) - \varphi(-r + it) \big)\, dt \right| \le \frac{2\pi e^{(\theta - \pi) r} \|f\|}{1 - e^{-2\pi r}} + \frac{2\pi e^{(-\theta - \pi) r} \|f\|}{1 - e^{-2\pi r}} \to 0.$$

Therefore, we have

$$f(0) = \int_{-\infty}^{\infty} e^{\theta t} \left(\frac{1}{2} \exp\left(-\frac{i\theta}{2} \right) f\left(t - \frac{i}{2} \right) + \frac{1}{2} \exp\left(\frac{i\theta}{2} \right) f\left(t + \frac{i}{2} \right) \right) \frac{dt}{\cosh(\pi t)}.$$

□

Suppose that x is a self-adjoint operator on a Hilbert space H and let

$$x = \int_{-\infty}^{\infty} x P(dx)$$

be a spectral decomposition.

Proposition 1.6
$\operatorname{dom} e^{ikx} = \operatorname{dom} e^{-(\operatorname{Im} k)x}$ for each complex number k.

Proof.

$$\begin{aligned}\operatorname{dom} e^{ikx} &= \left\{ \xi : \int_{-\infty}^{\infty} |e^{ikx}|^2 P_\xi(dx) < \infty \right\} \\ &= \left\{ \xi : \int_{-\infty}^{\infty} |e^{-(\operatorname{Im} k)x}|^2 P_\xi(dx) < \infty \right\} \\ &= \operatorname{dom} e^{-(\operatorname{Im} k)x}.\end{aligned}$$ □

Remark 1.3
$\operatorname{dom} e^{ikx} = H$ for each real number k.

Proposition 1.7
$\operatorname{dom} e^{ik_2 x}$ is a subset of $\operatorname{dom} e^{ik_1 x}$ for each complex numbers k_1 and k_2 such that $0 \le \operatorname{Im} k_1 \le \operatorname{Im} k_2$.

Proof.

$$\int_{-\infty}^{\infty} |e^{-(\operatorname{Im} k_1)x}|^2 P_\xi(dx) \le \int_{-\infty}^{\infty} \left(1 + e^{-(\operatorname{Im} k_2)x}\right)^2 P_\xi(dx).$$ □

Proposition 1.8
Suppose that k_- and k_+ are complex numbers such that

$$\operatorname{Im} k_- < \operatorname{Im} k_+.$$

Suppose that ξ is an element of $\operatorname{dom} e^{ik_- x} \cap \operatorname{dom} e^{ik_+ x}$. Then the mapping

$$k \mapsto e^{ikx}\xi$$

is holomorphic on $\operatorname{Im} k_- < \operatorname{Im} k < \operatorname{Im} k_+$ and bounded continuous on $\operatorname{Im} k_- \le \operatorname{Im} k \le \operatorname{Im} k_+$.

Proof. We first prove that the mapping is bounded continuous on the stripe.

$$\begin{aligned}\|e^{ikx}\xi\|^2 &= \int_{-\infty}^{\infty} |e^{ikx}|^2 P_\xi(dx) \\ &\le \int_{-\infty}^{\infty} \left(e^{-(\operatorname{Im} k_-)x} + e^{-(\operatorname{Im} k_+)x}\right)^2 P_\xi(dx) < \infty.\end{aligned}$$

Since

$$\begin{aligned}|e^{ik'x} - e^{ikx}|^2 &\le \big(|e^{ik'x}| + |e^{ikx}|\big)^2 \\ &= \big(e^{-(\operatorname{Im} k')x} + e^{-(\operatorname{Im} k)x}\big)^2 \\ &\le 4\big(e^{-(\operatorname{Im} k_-)x} + e^{-(\operatorname{Im} k_+)x}\big)^2,\end{aligned}$$

we have

$$\lim_{k' \to k} \|e^{ik'x}\xi - e^{ikx}\xi\|^2 = \lim_{k' \to k} \int_{-\infty}^{\infty} |e^{ik'x} - e^{ikx}|^2 P_\xi(dx) = 0.$$

Suppose that η is any element of H. Then we have

$$(e^{ikx}\xi, \eta) = \int_{-\infty}^{\infty} e^{ikx} P_{\xi,\eta}(dx).$$

We have

$$\begin{aligned}|\frac{\partial e^{ikx}}{\partial k}| &= |ixe^{ikx}| \\ &= |x|e^{-(\operatorname{Im} k)x} \\ &\le |x|(e^{-m_- x} + e^{-m_+ x})\end{aligned}$$

for $\operatorname{Im} k_- < m_- \le \operatorname{Im} k \le m_+ < \operatorname{Im} k_+$ and we have

$$\int_{-\infty}^{\infty} |x|e^{-mx} |P_{\xi,\eta}|(dx) \le \left(\int_{-\infty}^{\infty} |x|^2 e^{-2mx} P_\xi(dx)\right)^{1/2} \|\eta\|$$

for $\operatorname{Im} k_- < m < \operatorname{Im} k_+$. Since

$$\lim_{|x| \to \infty} \frac{|x|e^{-mx}}{e^{-(\operatorname{Im} k_-)x} + e^{-(\operatorname{Im} k_+)x}} = 0,$$

we have

$$\int_{-\infty}^{\infty} |x|^2 e^{-2mx} P_\xi(dx) < \infty$$

and we have

$$\int_{-\infty}^{\infty} |x|(e^{-m_- x} + e^{-m_+ x})|P_{\xi,\eta}|(dx) < \infty. \quad \square$$

Suppose that a is a bounded operator on H and let x_0 be a real number. Then the bounded operator

$$f(x_0) = \int_{-\infty}^{\infty} e^{ikx_0} e^{-ikx} a e^{ikx} \frac{dk}{\cosh(\pi k)}$$

is Gelfand-Pettis integrable with respect to the σ-weak topology. Suppose that ξ and η are elements of

$$\operatorname{dom} \exp\left(\frac{x}{2}\right) \cap \operatorname{dom} \exp\left(-\frac{x}{2}\right).$$

Then the function
$$k \mapsto e^{ikx_0}(ae^{ikx}\xi, e^{i\bar{k}x}\eta)$$
is holomorphic on $|\mathrm{Im}\, k| < 2^{-1}$ and bounded continuous on $|\mathrm{Im}\, k| \leq 2^{-1}$. By Theorem 1.1, we have

$$\begin{aligned}(a\xi,\eta) = \frac{1}{2}\exp\Big(\frac{x_0}{2}\Big)\Big(f(x_0)\exp\Big(\frac{x}{2}\Big)\xi, \exp\Big(-\frac{x}{2}\Big)\eta\Big) \\ + \frac{1}{2}\exp\Big(-\frac{x_0}{2}\Big)\Big(f(x_0)\exp\Big(-\frac{x}{2}\Big)\xi, \exp\Big(\frac{x}{2}\Big)\eta\Big).\end{aligned}$$

Suppose that b is any bounded operator on H such that

$$\begin{aligned}(a\xi,\eta) = \frac{1}{2}\exp\Big(\frac{x_0}{2}\Big)\Big(b\exp\Big(\frac{x}{2}\Big)\xi, \exp\Big(-\frac{x}{2}\Big)\eta\Big) \\ + \frac{1}{2}\exp\Big(-\frac{x_0}{2}\Big)\Big(b\exp\Big(-\frac{x}{2}\Big)\xi, \exp\Big(\frac{x}{2}\Big)\eta\Big)\end{aligned}$$

for each elements ξ and η of

$$\mathrm{dom}\exp\Big(\frac{x}{2}\Big) \cap \mathrm{dom}\exp\Big(-\frac{x}{2}\Big) = \mathrm{dom}\Big(\int_{-\infty}^{\infty}\Big(\exp\Big(\frac{x}{2}\Big) + \exp\Big(-\frac{x}{2}\Big)\Big)P(dx)\Big).$$

Then the function
$$k \mapsto e^{ikx_0}(be^{ikx}\xi, e^{i\bar{k}x}\eta)$$
is holomorphic on $|\mathrm{Im}\, k| < 2^{-1}$ and bounded continuous on $|\mathrm{Im}\, k| \leq 2^{-1}$. We have
$$(b\xi,\eta) = \int_{-\infty}^{\infty} e^{ikx_0}(ae^{ikx}\xi, e^{ikx}\eta)\frac{dk}{\cosh(\pi k)} = \big(f(x_0)\xi,\eta\big)$$
by Theorem 1.1. Then we have
$$b = \int_{-\infty}^{\infty} e^{ikx_0}e^{-ikx}ae^{ikx}\frac{dk}{\cosh(\pi k)} = f(x_0).$$

1.5 Tomita's theorem for a cyclic and separating vector

According to Section 1.4, we have
$$x'_\omega = \int_{-\infty}^{\infty} e^{i\omega t}e^{-iht}JxJe^{iht}\frac{dt}{\cosh(\pi t)}.$$

Proposition 1.9
$e^{-iht}JNJe^{iht}$ is a subset of N'.

Proof. Suppose that y is any element of N. Then we have
$$0 = \int_{-\infty}^{\infty} e^{i\omega t}[e^{-iht}JxJe^{iht}, y]\frac{dt}{\cosh(\pi t)}.$$

We have

$$[e^{-iht}JxJe^{iht}, y] = 0$$

by the uniqueness theorem of Fourier transforms. □

We have the following theorem by the relation $F = S^*$.

THEOREM 1.2 (Tomita's theorem for a cyclic and separating vector)

$$JNJ = N'$$

and

$$e^{iht}Ne^{-iht} = N, \qquad e^{iht}N'e^{-iht} = N'$$

for each real number t.

2 Hilbert Algebras

2.1 Locally compact groups

Suppose that Δ is the modular function on a locally compact group G.

There exists a unique Radon measure up to multiplication by a positive constant such that

$$d\mu(g) = \sqrt{\Delta(g)}\, dg, \qquad d\nu(g) = \frac{1}{\sqrt{\Delta(g)}}\, dg = \sqrt{\Delta(g^{-1})}\, dg$$

are a left Haar measure and a right Haar measure, respectively.

REMARK 2.1
$\mu(S^{-1}) = \nu(S)$ for any Borel set S.

We define $|S|$ to be the measure of a Borel set S. Then we have

$$\begin{aligned} |S| &= \int_S \sqrt{\Delta(g^{-1})}\, d\mu(g) \\ &= \int_{S^{-1}} \sqrt{\Delta(g)}\, d\nu(g) = |S^{-1}| \end{aligned}$$

for any Borel set S.

Suppose that ξ and η are elements of $C_C(G)$. Then the support of the continuous function

$$(g, h) \mapsto \zeta(g, h) = \xi(h)\eta(h^{-1}g)$$

is a subset of $(\operatorname{supp}\xi)(\operatorname{supp}\eta) \times \operatorname{supp}\xi$ and the function

$$\begin{aligned} g \mapsto (\xi * \eta)(g) &= \int \zeta(g, h)\, d\mu(h) \\ &= \int \xi(h)\eta(h^{-1}g)\, d\mu(h) \end{aligned}$$

is an element of $C_C(G)$. We have

$$(\xi * \eta)(g) = \int \xi(h^{-1})\eta(hg)\, d\nu(h)$$
$$= \int \xi(gh^{-1})\eta(h)\, d\nu(h).$$

Proposition 2.1
$C_C(G)$ is a complex algebra.

Proof. The mapping

$$(\xi, \eta) \mapsto \xi * \eta \tag{1}$$

is bilinear and we proceed to show that (1) is associative. We have

$$\big((\xi * \eta) * \zeta\big)(g) = \int (\xi * \eta)(h)\zeta(h^{-1}g)\, d\mu(h)$$
$$= \int \xi(h_1)\eta(h_1^{-1}h_2)\zeta(h_2^{-1}g)\, d(\mu \times \mu)(h_1, h_2).$$

Since

$$\int \eta(h_1^{-1}h_2)\zeta(h_2^{-1}g)\, d\mu(h_2) = \int \eta(h_2)\zeta(h_2^{-1}h_1^{-1}g)\, d\mu(h_2)$$
$$= (\eta * \zeta)(h_1^{-1}g),$$

we have

$$\big((\xi * \eta) * \zeta\big)(g) = \int \xi(h)(\eta * \zeta)(h^{-1}g)\, d\mu(h) = \big(\xi * (\eta * \zeta)\big)(g). \quad \square$$

We define a conjugate linear isometry J on $L^2(G)$ by

$$(J\xi)(g) = \overline{\xi(g^{-1})}$$

and we define a closed operator S on $L^2(G)$ by

$$S = J\sqrt{\Delta}.$$

Then we have

$$S = \sqrt{\Delta}^{-1} J = S^{-1}.$$

Since

$$\operatorname{dom} S = L^2\Big(\big(1 + \Delta(g)\big)\, dg\Big),$$

the algebra $C_C(G)$ is an invariant core for S.

Suppose that ξ and η are elements of $C_C(G)$. Then we have

$$\begin{aligned}
J(\xi * \eta)(g) &= \overline{(\xi * \eta)(g^{-1})} \\
&= \overline{\int \xi(h)\eta(h^{-1}g^{-1})\, d\mu(h)} \\
&= \overline{\int \xi(g^{-1}h)\eta(h^{-1})\, d\mu(h)} \\
&= \int (J\xi)(h^{-1}g)(J\eta)(h)\, d\mu(h) \\
&= \big((J\eta) * (J\xi)\big)(g)
\end{aligned}$$

and we have $J(\xi * \eta) = (J\eta) * (J\xi)$.

Proposition 2.2
S is an involution on $C_C(G)$.

Proof.

$$\begin{aligned}
S(\xi * \eta)(g) &= \sqrt{\Delta(g^{-1})} \int (J\eta)(h)(J\xi)(h^{-1}g)\, d\mu(h) \\
&= \int (S\eta)(h)(S\xi)(h^{-1}g)\, d\mu(h) \\
&= \big((S\eta) * (S\xi)\big)(g).
\end{aligned}$$

□

Suppose that ξ and η are elements of $C_C(G)$. Then we have

$$\begin{aligned}
\|\xi * \eta\|_2^2 &= \int \Big| \int \xi(h)\eta(h^{-1}g)\, d\mu(h)\Big|^2\, dg \\
&\le \int |\xi(h_1)||\xi(h_2)| \int |\eta(h_1^{-1}g)||\eta(h_2^{-1}g)|\, dg\, d(\mu \times \mu)(h_1, h_2).
\end{aligned}$$

Since

$$\begin{aligned}
\int |\eta(h^{-1}g)|^2\, dg &= \int |\eta(h^{-1}g)|^2 \sqrt{\Delta(g^{-1})}\, d\mu(g) \\
&= \int |\eta(g)|^2 \sqrt{\Delta(h^{-1})}\, dg \\
&= \frac{\|\eta\|_2^2}{\sqrt{\Delta(h)}},
\end{aligned}$$

we have

$$\begin{aligned}
\|\xi * \eta\|_2 &\le \int |\xi(h)| \frac{\|\eta\|_2}{\sqrt[4]{\Delta(h)}}\, d\mu(h) \\
&= \|\sqrt[4]{\Delta}\xi\|_1 \|\eta\|_2.
\end{aligned}$$

Proposition 2.3

The mapping

$$\eta \mapsto \pi(\xi)\eta = \xi * \eta$$

is bounded on $C_C(G)$ and we have $\|\pi(\xi)\| \leq \|\sqrt[4]{\Delta}\xi\|_1$.

Suppose that K is a compact neighborhood of e. Then there exists an element ξ_K of $C_C^+(G)$ such that $\operatorname{supp}\xi_K$ is a subset of K and

$$\int \xi_K(g)\,d\mu(g) = 1.$$

The set of compact neighborhoods of e is a directed set by reverse inclusion. Suppose that η is any element of $C_C(G)$. Then we have

$$\begin{aligned}|(\xi_K * \eta - \eta)(g)| &= \Big|\int \xi_K(h)\big(\eta(h^{-1}g) - \eta(g)\big)\,d\mu(h)\Big| \\ &\leq \int \xi_K(h)|\eta(h^{-1}g) - \eta(g)|\,d\mu(h).\end{aligned}$$

There exists a compact and symmetric neighborhood K_ε of e such that

$$\max_{g\in G}|\eta(h^{-1}g) - \eta(g)| < \varepsilon$$

provided that h is an element of K_ε. The net $(\xi_K * \eta)_K$ converges to η uniformly and we have $\lim_K \|\xi_K * \eta - \eta\|_2 = 0$.

Theorem 2.1

$C_C(G)$ is a Hilbert algebra.

Proof.

$$\begin{aligned}(\xi * \eta, \zeta) &= \int (\xi * \eta)(g)\overline{\zeta(g)}\,dg \\ &= \int \xi(h)\eta(h^{-1}g)\sqrt{\Delta(g^{-1})}\,d(\mu \times \mu)(g,h) \\ &= \int \xi(h)\eta(g)\overline{\zeta(hg)}\,d(g,h).\end{aligned}$$

$$\begin{aligned}\big(\eta, (S\xi) * \zeta\big) &= \int \eta(g)\overline{\int (S\xi)(h)\zeta(h^{-1}g)\,d\mu(h)}\,dg \\ &= \int \eta(g)\overline{\int (S\xi)(h^{-1})\zeta(hg)\,d\nu(h)}\,dg \\ &= \int \eta(g)\int \xi(h)\overline{\zeta(hg)}\,dh\,dg \\ &= \int \xi(h)\eta(g)\overline{\zeta(hg)}\,d(g,h).\end{aligned}$$

□

References

[1] Masamichi Takesaki. *Sayōsokan no kōzō.* Iwanami Shoten, Tokyo, 1983.