Tensor Products

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Definition 1

Suppose that R is a commutative ring with identity. A tensor product of unital R-modules M_1 and M_2 is a unital R-module $M_1 \otimes M_2$ with a bilinear mapping

$$M_1 \times M_2 \to M_1 \otimes M_2, \qquad (m_1, m_2) \mapsto \tau(m_1, m_2) = m_1 \otimes m_2 \tag{1}$$

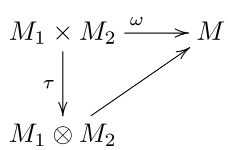
such that for any unital R-module M and for any bilinear mapping

$$M_1 \times M_2 \to M, \qquad (m_1, m_2) \mapsto \omega(m_1, m_2) \tag{2}$$

there exists a unique linear mapping

$$M_1 \otimes M_2 \to M \tag{3}$$

such that the following diagram commutes.



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Definition 2

Suppose that M is a right module over a ring R and let G be an abelian group. We define

$$R \times \hom(M, G) \to \hom(M, G), \qquad (r, f) \mapsto rf = \big[m \mapsto f(mr)\big]. \tag{5}$$

Proposition 1

Suppose that M is a right module over a ring R and let G be an abelian group. The abelian group $\hom(M, G)$ is a left module over the ring R.

Proof. We have the following.

1. We have
$$r(f_1 + f_2) = rf_1 + rf_2 \tag{6}$$
for any element (r, f) of the set $R \times \hom(M, G)^2$ since we have
$$\begin{aligned} \big(r(f_1 + f_2)\big)(m) &= (f_1 + f_2)(mr) \\ &= f_1(mr) + f_2(mr) \\ &= (rf_1)(m) + (rf_2)(m) = (rf_1 + rf_2)(m) \end{aligned} \tag{7}$$
for any element m.

2. We have

$$(r_1 + r_2)f = r_1 f + r_2 f \tag{8}$$

for any element (r, f) of the set $R^2 \times \hom(M, G)$ since we have

$$\begin{aligned}\big((r_1 + r_2)f\big)(m) &= f\big(m(r_1 + r_2)\big)\\ &= f(mr_1 + mr_2)\\ &= f(mr_1) + f(mr_2)\\ &= (r_1 f)(m) + (r_2 f)(m) = (r_1 f + r_2 f)(m)\end{aligned} \tag{9}$$

for any element m.

3. We have

$$(r_1 r_2)f = r_1(r_2 f) \tag{10}$$

for any element (r, f) of the set $R^2 \times \hom(M, G)$ since we have

$$\begin{aligned}\big((r_1 r_2)f\big)(m) &= f\big(m(r_1 r_2)\big)\\ &= f\big((mr_1)r_2\big)\\ &= (r_2 f)(mr_1) = \big(r_1(r_2 f)\big)(m)\end{aligned} \tag{11}$$

for any element m. □

Definition 3

A mapping of an abelian group into an abelian group is said to be additive if it is a homomorphism of groups.

Definition 4

Suppose that M is a right module and N is a left module over a ring R. A biadditive mapping ω of the product $M \times N$ into an abelian group is called a bihomomorphism if we have

$$\omega(mr, n) = \omega(m, rn) \tag{12}$$

for any element (r, m, n) of the set $R \times M \times N$.

Proposition 2

A right module over a ring R is a $(\mathbb{Z}, R)$-bimodule.

Proposition 3

A module over a commutative ring R is an (R, R)-bimodule.

Proposition 4

An abelian group is a $(\mathbb{Z}, \mathbb{Z})$-bimodule.

Definition 5

Suppose that R_0, R_1, R_2 are rings and let M_1 and M_2 be an (R_0, R_1)-bimodule and an (R_1, R_2)-bimodule respectively. A mapping ω of the product $M_1 \times M_2$ into an (R_0, R_2)-bimodule is called a bihomomorphism if it satisfies the following.

1. The mapping
$$m_1 \mapsto \omega(m_1, m_2) \tag{13}$$
is a homomorphism of left modules over the ring R_0 for any element m_2.

2. We have
$$\omega(m_1 r_1, m_2) = \omega(m_1, r_1 m_2) \tag{14}$$
for any element (r_1, m) of the set $R_1 \times (M_1 \times M_2)$.

3. The mapping
$$m_2 \mapsto \omega(m_1, m_2) \tag{15}$$
is a homomorphism of right modules over the ring R_2 for any element m_1.

The definition is compatible with Definition 4. See Proposition 2 and 4.

Definition 6
A homomorphism of bimodules is a homomorphism of left and right modules.

Definition 7
Suppose that
$$R_0, \qquad R_1, \qquad \dots, \qquad R_{n-1}, \qquad R_n \tag{16}$$
are rings and let
$$M_1, \qquad \dots, \qquad M_n \tag{17}$$
be an (R_0, R_1)-bimodule, ..., an (R_{n-1}, R_n)-bimodule respectively. A multiadditive mapping ω of the product $M_1 \times \dots \times M_n$ into an (R_0, R_n)-bimodule is called a multihomomorphism if it satisfies the following.

1. The mapping
$$m_1 \mapsto \omega(m_1, m_2, \dots, m_n) \tag{18}$$
is a homomorphism of left modules over the ring R_0 for any element $(m_2, \dots, m_n)$.

2. We have
$$\begin{aligned}\omega(m_1, \dots, m_{k-1}, m_k r_k, m_{k+1}, m_{k+2}, \dots, m_n)\\ = \omega(m_1, \dots, m_{k-1}, m_k, r_k m_{k+1}, m_{k+2}, \dots, m_n)\end{aligned} \tag{19}$$
for any element (r_k, m) of the set $R_k \times (M_1 \times \dots \times M_n)$ for any positive integer $k < n$.

3. The mapping
$$m_n \mapsto \omega(m_1, \dots, m_{n-1}, m_n) \tag{20}$$
is a homomorphism of right modules over the ring R_n for any element $(m_1, \dots, m_{n-1})$.

Remark 1

Suppose that R is a ring. Any mapping of the product $\{()\}$ into an (R,R)-bimodule is a multihomomorphism.

Remark 2

Suppose that R and S are rings and let M and N are (R,S)-bimodules. A mapping of the bimodule M into the bimodule N is a multihomomorphism if and only if it is a homomorphism of bimodules.

Proposition 5

Suppose that M is a right module and N is a left module over a ring R and let G be an abelian group. The following are equivalent for a mapping ω of the product $M\times N$ into the abelian group G.

1. The mapping ω is a bihomomorphism.
2. The mapping

$$N\to G, \qquad n\mapsto \omega(m,n) \tag{21}$$

is a homomorphism of groups for any element m and the mapping

$$M\to \hom(N,G), \qquad m\mapsto \big[n\mapsto \omega(m,n)\big] \tag{22}$$

is a homomorphism of right modules over the ring R.

3. The mapping

$$M\to G, \qquad m\mapsto \omega(m,n) \tag{23}$$

is a homomorphism of groups for any element n and the mapping

$$N\to \hom(M,G), \qquad n\mapsto \big[m\mapsto \omega(m,n)\big] \tag{24}$$

is a homomorphism of left modules over the ring R.

Proof. Suppose that the mapping ω is a bihomomorphism. The mapping

$$N\to G, \qquad n\mapsto \omega(m,n) \tag{25}$$

is a homomorphism of groups for any element m since we have

$$\omega(m,n_1+n_2)=\omega(m,n_1)+\omega(m,n_2) \tag{26}$$

for any element n of the set N^2. The mapping

$$M\to \hom(N,G), \qquad m\mapsto \big[n\mapsto \omega(m,n)\big] \tag{27}$$

is a homomorphism of right modules over the ring R since we have

$$\omega(m_1+m_2,n)=\omega(m_1,n)+\omega(m_2,n) \tag{28}$$

for any element (m, n) of the set $M^2 \times N$ and we have

$$\omega(mr, n) = \omega(m, rn) \tag{29}$$

for any element (m, r, n) of the set $M \times R \times N$.

Suppose that the mapping

$$N \to G, \qquad n \mapsto \omega(m, n) \tag{30}$$

is a homomorphism of groups for any element m and the mapping

$$M \to \hom(N, G), \qquad m \mapsto \left[n \mapsto \omega(m, n)\right] \tag{31}$$

is a homomorphism of right modules over the ring R. The mapping

$$m \mapsto \omega(m, n) \tag{32}$$

is a homomorphism of groups of the right module M into the abelian group G for any element n of the left module N and we have

$$\omega(mr, n) = \omega(m, rn) \tag{33}$$

for any element (m, r, n) of the set $M \times R \times N$ since the mapping

$$M \to \hom(N, G), \qquad m \mapsto \left[n \mapsto \omega(m, n)\right] \tag{34}$$

is a homomorphism of right modules over the ring R. □

PROPOSITION 6

Suppose that M is a right module and N is a left module over a ring R and let G be an abelian group. Suppose that ω is a bihomomorphism of the product $M \times N$ into the abelian group G and let f be a homomorphism of groups of the abelian group G into an abelian group. The mapping $f \circ \omega$ is a bihomomorphism.

DEFINITION 8

A tensor product of a right module M and a left module N over a ring R is an abelian group $M \otimes N$ with a bihomomorphism

$$M \times N \to M \otimes N, \qquad (m, n) \mapsto \tau(m, n) = m \otimes n \tag{35}$$

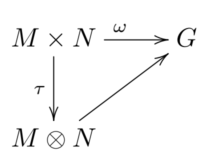
such that for any abelian group G and for any bihomomorphism

$$M \times N \to G, \qquad (m, n) \mapsto \omega(m, n) \tag{36}$$

there exists a unique homomorphism of groups

$$M \otimes N \to G \tag{37}$$

such that the following diagram commutes.



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Proposition 7

Suppose that

$$M \otimes_1 N \xleftarrow{\tau_1} M \times N \xrightarrow{\tau_2} M \otimes_2 N \tag{39}$$

are tensor products of a right module M and a left module N over a ring R. There exists a unique isomorphism of groups

$$M \otimes_1 N \leftrightarrow M \otimes_2 N \tag{40}$$

such that the following diagram commutes.

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau_2} & M \otimes_2 N \\ {\scriptstyle \tau_1}\downarrow & \nearrow\!\!\!\!\swarrow & \\ M \otimes_1 N & & \end{array} \tag{41}$$

Proof. There exist unique homomorphisms of groups f_1 and f_2 such that the following diagrams commute.

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau_2} & M \otimes_2 N \\ {\scriptstyle \tau_1}\downarrow & \nearrow {\scriptstyle f_1} & \\ M \otimes_1 N & & \end{array}, \qquad \begin{array}{ccc} M \times N & \xrightarrow{\tau_2} & M \otimes_2 N \\ {\scriptstyle \tau_1}\downarrow & \swarrow {\scriptstyle f_2} & \\ M \otimes_1 N & & \end{array} \tag{42}$$

The following diagrams commute.

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau_1} & M \otimes_1 N \\ {\scriptstyle \tau_1}\downarrow & \nearrow {\scriptstyle f_2 \circ f_1} & \\ M \otimes_1 N & & \end{array}, \qquad \begin{array}{ccc} M \times N & \xrightarrow{\tau_2} & M \otimes_2 N \\ {\scriptstyle \tau_2}\downarrow & \nearrow {\scriptstyle f_1 \circ f_2} & \\ M \otimes_2 N & & \end{array} \tag{43}$$

We have $f_2 \circ f_1 = 1$ and we have $f_1 \circ f_2 = 1$. □

Theorem 1

Suppose that M is a right module and N is a left module over a ring R. There exists a unique tensor product of the right module M and the left module N.

Proof. We write $\bar{\tau}$ for the canonical epimorphism of the free abelian group $\mathbb{Z}^{\oplus(M\times N)}$ onto the quotient group by the subgroup generated by the subset

$$\begin{aligned} &\big\{\, (mr,n) - (m,rn) : (m,r,n) \in M \times R \times N \,\big\} \\ &\quad \cup \big\{\, (m_1+m_2,n) - (m_1,n) - (m_2,n) : (m_1,m_2,n) \in M \times M \times N \,\big\} \\ &\quad \cup \big\{\, (m,n_1+n_2) - (m,n_1) - (m,n_2) : (m,n_1,n_2) \in M \times N \times N \,\big\}. \end{aligned}$$

The restriction τ of the canonical epimorphism $\bar{\tau}$ to the product $M \times N$ is a bihomomorphism. Suppose that G is an abelian group and let ω be a bihomomorphism of the product $M\times N$ into the abelian group G. The bihomomorphism

ω extends uniquely to the homomorphism of groups of the free abelian group $\mathbb{Z}^{\oplus(M\times N)}$.

$$\begin{array}{ccc} M\times N & \longrightarrow & G \\ \downarrow & \nearrow_{\omega} & \\ \mathbb{Z}^{\oplus(M\times N)} & & \end{array} \tag{44}$$

The homomorphism ω factors uniquely through the canonical epimorphism $\bar{\tau}$. $\square$

Proposition 8

Suppose that M is a right module and N is a left module over a ring R. The abelian group $M\otimes N$ is the monoid generated by the subset

$$\big\{\, m\otimes n : (m,n)\in M\times N \,\big\}. \tag{45}$$

Proposition 9

Suppose that M_1 and N_1 are right modules and M_2 and N_2 are left modules over a ring R. Suppose that f_1 is a homomorphism of right modules over the ring R of the right module M_1 into the right module N_1. Suppose that f_2 is a homomorphism of left modules over the ring R of the left module M_2 into the left module N_2. There exists a unique homomorphism of groups of the abelian group $M_1\otimes M_2$ into the abelian group $N_1\otimes N_2$ such that the following diagram commutes.

$$\begin{array}{ccc} M_1\times M_2 & \xrightarrow{f_1\times f_2} & N_1\times N_2 \\ \otimes\Big\downarrow & & \Big\downarrow\otimes \\ M_1\otimes M_2 & \longrightarrow & N_1\otimes N_2 \end{array} \tag{46}$$

Proposition 10

Suppose that M and N are abelian groups and let G be an abelian group. A mapping ω of the product $M\times N$ into the abelian group G is a bihomomorphism if and only if it is a bilinear mapping.

Proposition 11

Suppose that M and N are unital modules over a commutative ring R. The (R,R)-bimodule $M\otimes N$ with the bihomomorphism

$$M\times N\to M\otimes N, \qquad (m,n)\mapsto m\otimes n \tag{47}$$

is a tensor product in the sense of Definition 1.

Definition 9

Suppose that

$$R_0, \qquad R_1, \qquad \ldots, \qquad R_{n-1}, \qquad R_n \tag{48}$$

are rings and let

$$M_1, \qquad \ldots, \qquad M_n \tag{49}$$

be an (R_0,R_1)-bimodule, ..., an (R_{n-1},R_n)-bimodule respectively. A tensor product of the bimodules M_1, ..., M_n is an (R_0,R_n)-bimodule $\bigotimes_{k=1}^n M_k$ with a multihomomorphism

$$\prod_{k=1}^n M_k \to \bigotimes_{k=1}^n M_k, \qquad m \mapsto \tau(m) = \bigotimes_{k=1}^n m_k \tag{50}$$

such that for any (R_0,R_n)-bimodule M and for any multihomomorphism

$$\prod_{k=1}^n M_k \to M, \qquad m \mapsto \omega(m) \tag{51}$$

there exists a unique homomorphism of bimodules

$$\bigotimes_{k=1}^n M_k \to M \tag{52}$$

such that the following diagram commutes.

$$\begin{array}{ccc} \prod_{k=1}^n M_k & \xrightarrow{\ \omega\ } & M \\ {\scriptstyle\tau}\downarrow & \nearrow & \\ \bigotimes_{k=1}^n M_k & & \end{array} \tag{53}$$

Proposition 12

Suppose that

$$R_0, \qquad R_1, \qquad \ldots, \qquad R_{n-1}, \qquad R_n \tag{54}$$

are rings and let

$$M_1, \qquad \ldots, \qquad M_n \tag{55}$$

be an (R_0,R_1)-bimodule, ..., an (R_{n-1},R_n)-bimodule respectively. Suppose that

$$M_1 \otimes_1 \cdots \otimes_1 M_n \xleftarrow{\ \tau_1\ } M_1 \times \cdots \times M_n \xrightarrow{\ \tau_2\ } M_1 \otimes_2 \cdots \otimes_2 M_n \tag{56}$$

are tensor products of the bimodules M_1, ..., M_n. There exists a unique isomorphism of bimodules

$$M_1 \otimes_1 \cdots \otimes_1 M_n \leftrightarrow M_1 \otimes_2 \cdots \otimes_2 M_n \tag{57}$$

such that the following diagram commutes.

$$\begin{array}{ccc} M_1 \times \cdots \times M_n & \xrightarrow{\ \tau_2\ } & M_1 \otimes_2 \cdots \otimes_2 M_n \\ {\scriptstyle\tau_1}\downarrow & \swarrow\!\!\nearrow & \\ M_1 \otimes_1 \cdots \otimes_1 M_n & & \end{array} \tag{58}$$

Proof. There exist unique homomorphisms of bimodules f_1 and f_2 such that the following diagrams commute.

$$\begin{array}{ccc} M_1 \times \cdots \times M_n & \xrightarrow{\tau_2} & M_1 \otimes_2 \cdots \otimes_2 M_n \\ \downarrow{\scriptstyle \tau_1} & \nearrow{\scriptstyle f_1} & \\ M_1 \otimes_1 \cdots \otimes_1 M_n & & \end{array} \tag{59}$$

$$\begin{array}{ccc} M_1 \times \cdots \times M_n & \xrightarrow{\tau_2} & M_1 \otimes_2 \cdots \otimes_2 M_n \\ \downarrow{\scriptstyle \tau_1} & \swarrow{\scriptstyle f_2} & \\ M_1 \otimes_1 \cdots \otimes_1 M_n & & \end{array} \tag{60}$$

The following diagrams commute.

$$\begin{array}{ccc} M_1 \times \cdots \times M_n & \xrightarrow{\tau_1} & M_1 \otimes_1 \cdots \otimes_1 M_n \\ \downarrow{\scriptstyle \tau_1} & \nearrow{\scriptstyle f_2 \circ f_1} & \\ M_1 \otimes_1 \cdots \otimes_1 M_n & & \end{array} \tag{61}$$

$$\begin{array}{ccc} M_1 \times \cdots \times M_n & \xrightarrow{\tau_2} & M_1 \otimes_2 \cdots \otimes_2 M_n \\ \downarrow{\scriptstyle \tau_2} & \nearrow{\scriptstyle f_1 \circ f_2} & \\ M_1 \otimes_2 \cdots \otimes_2 M_n & & \end{array} \tag{62}$$

We have $f_2 \circ f_1 = 1$ and we have $f_1 \circ f_2 = 1$. □

Proposition 13

Suppose that R_0, R_1, R_2 are rings and let M_1 and M_2 be an (R_0, R_1)-bimodule and an (R_1, R_2)-bimodule respectively.

1. The abelian group $M_1 \otimes M_2$ is an (R_0, R_2)-bimodule.
2. The bihomomorphism into the abelian group

$$M_1 \times M_2 \to M_1 \otimes M_2, \qquad (m_1, m_2) \mapsto m_1 \otimes m_2 \tag{63}$$

   is a bihomomorphism into the (R_0, R_2)-bimodule.

3. The (R_0, R_2)-bimodule $M_1 \otimes M_2$ with the bihomomorphism

$$M_1 \times M_2 \to M_1 \otimes M_2, \qquad (m_1, m_2) \mapsto m_1 \otimes m_2 \tag{64}$$

   is a tensor product in the sense of Definition 9.

Proof. Suppose that M is an (R_0, R_2)-bimodule and let

$$M_1 \times M_2 \to M, \qquad (m_1, m_2) \mapsto \omega(m_1, m_2) \tag{65}$$

be a bihomomorphism. There exists a unique homomorphism of groups

$$M_1 \otimes M_2 \to M \tag{66}$$

such that the following diagram commutes since the bihomomorphism ω is a bihomomorphism into the abelian group.

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{\omega} & M \\ {\scriptstyle\otimes}\downarrow & \nearrow & \\ M_1 \otimes M_2 & & \end{array} \tag{67}$$

The homomorphism of groups (66) is a homomorphism of (R_0, R_2)-bimodules. □

PROPOSITION 14
Suppose that M_1 and N_1 are right modules and M_2 and N_2 are left modules over a ring R. There exists a unique bihomomorphism

$$\big(\hom_R(M_1, N_1) \otimes_{\mathbb{Z}} \hom_R(M_2, N_2)\big) \times (M_1 \otimes M_2) \to N_1 \otimes N_2, \quad (f, m) \mapsto fm \tag{68}$$

such that we have

$$(f_1 \otimes_{\mathbb{Z}} f_2)(m_1 \otimes m_2) = f_1(m_1) \otimes f_2(m_2) \tag{69}$$

for any element (f, m) of the set

$$\big(\hom_R(M_1, N_1) \times \hom_R(M_2, N_2)\big) \times (M_1 \times M_2). \tag{70}$$

REMARK 3
A ring R is an (R, R)-bimodule.

PROPOSITION 15
Suppose that R_0 and R_1 are rings and let M_1 be an (R_0, R_1)-bimodule.

1. We have an isomorphism of (R_0, R_1)-bimodule

$$R_0 \otimes M_1 = M_1, \qquad r_0 \otimes m_1 = r_0 m_1 \tag{71}$$

provided that the left module M_1 is unital.

2. We have an isomorphism of (R_0, R_1)-bimodule

$$M_1 \otimes R_1 = M_1, \qquad m_1 \otimes r_1 = m_1 r_1 \tag{72}$$

provided that the right module M_1 is unital.

Proof. There exists a unique homomorphism of (R_0, R_1)-bimodules f

$$R_0 \otimes M_1 \to M_1, \qquad r_0 \otimes m_1 \mapsto r_0 m_1 \tag{73}$$

since the mapping

$$R_0 \times M_1 \to M_1, \qquad (r_0, m_1) \mapsto r_0 m_1 \tag{74}$$

is a bihomomorphism. The homomorphism of groups

$$M_1 \to R_0 \otimes M_1, \qquad m_1 \mapsto g(m_1) = 1 \otimes m_1 \tag{75}$$

is a homomorphism of (R_0, R_1)-bimodules since we have

$$\begin{aligned} g(r_0 m_1) &= 1 \otimes (r_0 m_1) \\ &= r_0 \otimes m_1 \\ &= r_0(1 \otimes m_1) = r_0 g(m_1) \end{aligned} \tag{76}$$

and we have

$$\begin{aligned} g(m_1 r_1) &= 1 \otimes (m_1 r_1) \\ &= (1 \otimes m_1) r_1 = g(m_1) r_1 \end{aligned} \tag{77}$$

for any element (r, m_1) of the set $(R_0 \times R_1) \times M_1$. The homomorphism of (R_0, R_1)-bimodules g is an epimorphism and we have

$$(f \circ g)(m_1) = f(1 \otimes m_1) = m_1 \tag{78}$$

for any element m_1 since the left module M_1 is unital. □

PROPOSITION 16

Suppose that R_0, R_1, R_2, R_3 are rings and let M_1, M_2, M_3 be an (R_0, R_1)-bimodule, an (R_1, R_2)-bimodule, an (R_2, R_3)-bimodule respectively. There exists a unique isomorphism of (R_0, R_3)-bimodules

$$\begin{aligned} &(M_1 \otimes M_2) \otimes M_3 = M_1 \otimes (M_2 \otimes M_3), \\ &\qquad (m_1 \otimes m_2) \otimes m_3 = m_1 \otimes (m_2 \otimes m_3). \end{aligned} \tag{79}$$

Proof. The (R_0, R_3)-bimodule $(M_1 \otimes M_2) \otimes M_3$ is the monoid generated by the subset

$$\left\{ (m_1 \otimes m_2) \otimes m_3 : m \in M_1 \times M_2 \times M_3 \right\}. \tag{80}$$

There exists a unique mapping

$$(M_1 \otimes M_2) \times M_3 \to M_1 \otimes (M_2 \otimes M_3), \qquad (m_{12}, m_3) \mapsto \omega(m_{12}, m_3) \tag{81}$$

such that the mapping

$$M_1 \otimes M_2 \to M_1 \otimes (M_2 \otimes M_3), \qquad m_{12} \mapsto \omega(m_{12}, m_3) \tag{82}$$

is a homomorphism of left modules for any element m_3 and we have

$$\omega(m_1 \otimes m_2, m_3) = m_1 \otimes (m_2 \otimes m_3) \tag{83}$$

for any element m since the mapping

$$M_1 \times M_2 \to M_1 \otimes (M_2 \otimes M_3), \qquad (m_1, m_2) \mapsto m_1 \otimes (m_2 \otimes m_3) \tag{84}$$

is a bihomomorphism into the left module for any element m_3. There exists a unique homomorphism of (R_0, R_3)-bimodules

$$(M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3), \qquad (m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3) \tag{85}$$

since the mapping ω is a bihomomorphism into the (R_0, R_3)-bimodule. □

THEOREM 2

Suppose that

$$R_0, \qquad R_1, \qquad \dots, \qquad R_{n-1}, \qquad R_n \tag{86}$$

are rings and let

$$M_1, \qquad \dots, \qquad M_n \tag{87}$$

be an (R_0, R_1)-bimodule, …, an (R_{n-1}, R_n)-bimodule respectively.

1. There exists a unique tensor product of the bimodules M_1, …, M_n provided that the integer n is positive.
2. Suppose that the integer n is positive. We have

$$\bigotimes_{k=1}^{1} M_k = M_1, \qquad \bigotimes_{k=1}^{1} m_k = m_1 \tag{88}$$

and we have

$$\bigotimes_{k=1}^{n} M_k = (\bigotimes_{k=1}^{n-1} M_k) \otimes M_n, \qquad \bigotimes_{k=1}^{n} m_k = (\bigotimes_{k=1}^{n-1} m_k) \otimes m_n \tag{89}$$

provided that we have $n > 1$.
3. The (R_0, R_n)-bimodule $\bigotimes_{k=1}^{n} M_k$ is the monoid generated by the subset

$$\Big\{ \bigotimes_{k=1}^{n} m_k : m \in \prod_{k=1}^{n} M_k \Big\} \tag{90}$$

provided that the integer n is positive.

Proposition 17

Suppose that

$$R_0, \qquad R_1, \qquad \dots, \qquad R_{n-1}, \qquad R_n \tag{91}$$

are rings and let

$$M_1, \qquad \dots, \qquad M_n \tag{92}$$

be an (R_0, R_1)-bimodule, ..., an (R_{n-1}, R_n)-bimodule respectively.

1. Suppose that ω is a multihomomorphism of the product $\prod_{k=1}^n M_k$ into an (R_0, R_n)-bimodule M and let f be a homomorphism of (R_0, R_n)-bimodules of the (R_0, R_n)-bimodule M into an (R_0, R_n)-bimodule. The mapping $f \circ \omega$ is a multihomomorphism.

2. Suppose that M is an (R_0, R_n)-bimodule. The abelian group

$$\hom_{(R_0,R_n)}(\bigotimes_{k=1}^{n} M_k, M) \tag{93}$$

is the abelian group of multihomomorphisms of the product $\prod_{k=1}^n M_k$ into the (R_0, R_n)-bimodule M provided that the integer n is positive.

Proposition 18

Suppose that A is an algebra and M is a unital module over a commutative ring. The tensor product of the algebra A and the unital module M is an (A, A)-bimodule such that we have

$$x_1(x_2 \otimes m) = (x_1 x_2) \otimes m = (x_1 \otimes m)x_2 \tag{94}$$

for any element (x, m) of the set $A^2 \times M$.

Proposition 19

Suppose that A is an algebra with identity and M is a unital module over a commutative ring R.

1. The (A, A)-bimodule $A \otimes M$ is compatible with the R-module $A \otimes M$.

2. The (A, A)-bimodule $A \otimes M$ is unital.

Proof. 1. We have $(r1)(x \otimes m) = \big((r1)x\big) \otimes m = (rx) \otimes m = r(x \otimes m)$ for any element (r, x, m) of the set $R \times A \times M$.

2. We have $1(x \otimes m) = x \otimes m = (x \otimes m)1$ for any element (x, m) of the set $A \times M$. □

Theorem 3

Suppose that A is an algebra with identity and M_1 and M_2 are unital modules over a commutative ring R. We have an isomorphism of unital (A, A)-bimodules

$$A \otimes (M_1 \otimes M_2) = (A \otimes M_1) \otimes_A (A \otimes M_2), \quad (x_1 x_2) \otimes (m_1 \otimes m_2) = (x_1 \otimes m_1) \otimes_A (x_2 \otimes m_2). \tag{95}$$

Proof. There exists a unique bilinear mapping over the commutative ring R

$$\begin{aligned}&(A\otimes M_1)\times(A\otimes M_2)\to A\otimes(M_1\otimes M_2),\\&\qquad(x_1\otimes m_1,x_2\otimes m_2)\mapsto f(x_1\otimes m_1,x_2\otimes m_2)=(x_1x_2)\otimes(m_1\otimes m_2)\end{aligned}\tag{96}$$

since the mapping

$$\begin{aligned}&A\times M_1\times A\times M_2\to A\otimes(M_1\otimes M_2),\\&\qquad(x_1,m_1,x_2,m_2)\mapsto(x_1x_2)\otimes(m_1\otimes m_2)\end{aligned}\tag{97}$$

is bilinear over the commutative ring R. There exists a unique homomorphism of (A,A)-bimodules

$$\begin{aligned}&(A\otimes M_1)\otimes_A(A\otimes M_2)\to A\otimes(M_1\otimes M_2),\\&\qquad(x_1\otimes m_1)\otimes_A(x_2\otimes m_2)\mapsto(x_1x_2)\otimes(m_1\otimes m_2)\end{aligned}\tag{98}$$

since the bilinear mapping f is a bihomomorphism into the (A,A)-bimodule. We define a mapping

$$\begin{aligned}&A\times M_1\times M_2\to(A\otimes M_1)\otimes_A(A\otimes M_2),\\&\qquad(x,m_1,m_2)\mapsto g(x,m_1,m_2)=x\big((1\otimes m_1)\otimes_A(1\otimes m_2)\big)\\&\qquad\qquad=\big((1\otimes m_1)\otimes_A(1\otimes m_2)\big)x.\end{aligned}\tag{99}$$

The unital (A,A)-bimodule $(A\otimes M_1)\otimes_A(A\otimes M_2)$ is a unital module over the commutative ring R such that we have

$$\begin{aligned}g(rx_1x_2,m_1,m_2)&=r\big((x_1\otimes m_1)\otimes_A(x_2\otimes m_2)\big)\\&=\big((x_1\otimes m_1)\otimes_A(x_2\otimes m_2)\big)r\end{aligned}\tag{100}$$

for any element (r,x,m) of the set $R\times A^2\times(M_1\times M_2)$. There exists a unique homomorphism of modules over R

$$\begin{aligned}&A\otimes(M_1\otimes M_2)\to(A\otimes M_1)\otimes_A(A\otimes M_2),\\&\qquad x\otimes(m_1\otimes m_2)\mapsto g(x,m_1,m_2)\end{aligned}\tag{101}$$

since the mapping g is multilinear over the commutative ring R. The homomorphism (98) is an isomorphism since we have

$$\begin{aligned}(g\circ f)\big((x_1\otimes m_1)\otimes_A(x_2\otimes m_2)\big)&=g\big((x_1x_2)\otimes(m_1\otimes m_2)\big)\\&=(x_1\otimes m_1)\otimes_A(x_2\otimes m_2)\end{aligned}\tag{102}$$

for any element (x,m) of the set $A^2\times(M_1\times M_2)$ and we have

$$(f\circ g)\big(x\otimes(m_1\otimes m_2)\big)=x\otimes(m_1\otimes m_2)\tag{103}$$

for any element (x,m) of the set $A\times(M_1\times M_2)$. □

References

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