Schouten–Nijenhuis Bracket

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1 Tensor Products of Multilinear Mappings

Suppose that $A_1, \ldots, A_{m+n}, A$ are unital modules over a commutative ring R and let μ be a bilinear mapping on the unital module A.

PROPOSITION 1
The mapping

$$\hom_R(\bigotimes_{k=1}^{m} A_k, A) \times \hom_R(\bigotimes_{k=m+1}^{m+n} A_k, A) \to \hom_R(\bigotimes_{k=1}^{m+n} A_k, A),$$
$$(\xi, \eta) \mapsto \xi \otimes_\mu \eta = \mu \circ (\xi \otimes \eta)$$

is bilinear over the commutative ring R.

Suppose that (ξ, η) is an element of the product

$$\hom_R(\bigotimes_{k=1}^{m} A_k, A) \times \hom_R(\bigotimes_{k=m+1}^{m+n} A_k, A).$$

PROPOSITION 2
We have

$$(\xi \otimes_\mu \eta)(x_1, \ldots, x_{m+n}) = \mu\big(\xi(x_1, \ldots, x_m), \eta(x_{m+1}, \ldots, x_{m+n})\big)$$

for any elements $(x_1, \ldots, x_{m+n})$.

2 Schouten–Nijenhuis Bracket

DEFINITION 1
We define

$$S_{(m,n)} = \Big\{ \sigma \in S_{m+n} : \sigma(1) < \cdots < \sigma(m) \text{ and } \sigma(m+1) < \cdots < \sigma(m+n) \Big\}.$$

Definition 2

Suppose that A is a unital module over a commutative ring and let ξ and η be alternating mappings on the unital module A. We define

$$\begin{aligned}
&[\xi,\eta](x_0,\dots,x_{m+n})\\
&\qquad=\sum_{\sigma\in S_{(n+1,m)}}(\operatorname{sgn}\sigma)\xi\Big(\eta\big(x_{\sigma(0)},\dots,x_{\sigma(n)}\big),x_{\sigma(1+n)},\dots,x_{\sigma(m+n)}\Big)\\
&\quad-(-1)^{mn}\sum_{\sigma\in S_{(m+1,n)}}(\operatorname{sgn}\sigma)\eta\Big(\xi\big(x_{\sigma(0)},\dots,x_{\sigma(m)}\big),x_{\sigma(m+1)},\dots,x_{\sigma(m+n)}\Big)
\end{aligned}$$

for any elements $(x_0,\dots,x_{m+n})$.

Proposition 3

Suppose that A is a unital module over a commutative ring and let ξ and η be alternating mappings on the unital module A. The multilinear mapping

$$\begin{aligned}
&(x_0,\dots,x_{m+n})\\
&\qquad\mapsto\sum_{\sigma\in S_{(n+1,m)}}(\operatorname{sgn}\sigma)\xi\Big(\eta\big(x_{\sigma(0)},\dots,x_{\sigma(n)}\big),x_{\sigma(1+n)},\dots,x_{\sigma(m+n)}\Big)
\end{aligned}$$

is alternating.

Definition 3

Suppose that A is a unital module over a commutative ring and let ξ and η be alternating mappings on the unital module A. The alternating mapping $[\xi,\eta]$ is called the Schouten–Nijenhuis bracket of the alternating mappings ξ and η.

Proposition 4

Suppose that A is a unital module over a commutative ring and let μ be a bilinear mapping on the unital module A. Suppose that ξ and η are alternating multiderivations with respect to the bilinear mapping μ. The Schouten–Nijenhuis bracket $[\xi,\eta]$ is an alternating multiderivation with respect to the bilinear mapping μ.

Proof. It is sufficient to show that the linear mapping

$$x_0\mapsto[\xi,\eta](x_0,x_1,\dots,x_{m+n})$$

is a derivation with respect to the bilinear mapping μ for any elements $(x_0,\dots,x_{m+n})$ since the Schouten–Nijenhuis bracket $[\xi,\eta]$ is alternating. The linear mapping

$$\begin{aligned}
x_0\mapsto&\sum_{\substack{\sigma\in S_{(n+1,m)}\\ \sigma(0)\neq 0}}(\operatorname{sgn}\sigma)\xi\Big(\eta\big(x_{\sigma(0)},\dots,x_{\sigma(n)}\big),x_{\sigma(1+n)},\dots,x_{\sigma(m+n)}\Big)\\
&-(-1)^{mn}\sum_{\substack{\sigma\in S_{(m+1,n)}\\ \sigma(0)\neq 0}}(\operatorname{sgn}\sigma)\eta\Big(\xi\big(x_{\sigma(0)},\dots,x_{\sigma(m)}\big),x_{\sigma(m+1)},\dots,x_{\sigma(m+n)}\Big)
\end{aligned}$$

is a derivation with respect to the bilinear mapping μ by Proposition 15. The linear mapping

$$\begin{aligned} x_0 \mapsto \sum_{\substack{\sigma\in S_{(n+1,m)}\\ \sigma(0)=0}} &(\operatorname{sgn}\sigma)\xi\Big(\eta\big(x_{\sigma(0)},\dots,x_{\sigma(n)}\big),x_{\sigma(1+n)},\dots,x_{\sigma(m+n)}\Big)\\ &- (-1)^{mn} \sum_{\substack{\sigma\in S_{(m+1,n)}\\ \sigma(0)=0}} (\operatorname{sgn}\sigma)\eta\Big(\xi\big(x_{\sigma(0)},\dots,x_{\sigma(m)}\big),x_{\sigma(m+1)},\dots,x_{\sigma(m+n)}\Big)\\ = (-1)^{mn} \sum_{\sigma\in S_{(m,n)}} &(\operatorname{sgn}\sigma)\bigg(\xi\Big(\eta\big(x_0,x_{\sigma(m+1)},\dots,x_{\sigma(m+n)}\big),x_{\sigma(1)},\dots,x_{\sigma(m)}\Big)\\ &- \eta\Big(\xi\big(x_0,x_{\sigma(1)},\dots,x_{\sigma(m)}\big),x_{\sigma(m+1)},\dots,x_{\sigma(m+n)}\Big)\bigg) \end{aligned}$$

is a derivation with respect to the bilinear mapping μ by Proposition 17. □

Proposition 5

The Schouten–Nijenhuis bracket $[\xi,\eta]$ of smooth polyvector fields ξ and η on a smooth manifold is a smooth polyvector field on the smooth manifold.

Proposition 6

Suppose that U is an open submanifold of a smooth manifold X. The following diagram commutes.

$$\begin{array}{ccc} V^{m+1}(X)\times V^{n+1}(X) & \longrightarrow & V^{m+n+1}(X) \\ \downarrow & & \downarrow \\ V^{m+1}(U)\times V^{n+1}(U) & \longrightarrow & V^{m+n+1}(U) \end{array}, \qquad (\xi,\eta)\mapsto[\xi,\eta].$$

Proposition 7

Suppose that ξ is a linear mapping and let η be an alternating mapping on a unital module over a commutative ring. We have

$$[\xi,\eta] = \xi\circ\eta - \eta\circ\sum_{m=1}^{n}\delta^{\otimes(m-1)}\otimes\xi\otimes\delta^{\otimes(n-m)}.$$

Proposition 8

Suppose that S is a set and let A be a unital module over a commutative ring R. The set A^S is a unital module over the commutative ring R.

Proposition 9

Suppose that A_1, …, A_n, A are unital modules over a commutative ring R.

1. The set of multilinear mapping of the product $\prod_{m=1}^{n} A_m$ into the unital module A is a submodule of the unital module

$$A^{\prod_{m=1}^{n} A_m}.$$

2. The unital module $\hom_R(\bigotimes_{m=1}^n A_m, A)$ is isomorphic to the unital module of multilinear mappings of the product $\prod_{m=1}^n A_m$ into the unital module A.

Proposition 10
Suppose that A, B, C are unital modules over a commutative ring R. The mapping

$$\hom_R(B,C) \times \hom_R(A,B) \to \hom_R(A,C), \qquad (\xi,\eta) \mapsto \xi \circ \eta$$

is bilinear over the commutative ring R.

Proposition 11
Suppose that A is a unital module over a commutative ring R.

1. The unital module $\hom_R(A,A)$ is an algebra with identity over the commutative ring R.

2. The algebra $\hom_R(A,A)$ is called the endomorphism algebra of the unital module A.

Definition 4
Suppose that ξ is a linear mapping and let η be a multilinear mapping on a unital module over a commutative ring.

1. We define the multilinear mapping

$$[\xi,\eta] = \xi \circ \eta - \eta \circ \sum_{m=1}^{n} \delta^{\otimes(m-1)} \otimes \xi \otimes \delta^{\otimes(n-m)}.$$

2. The multilinear mapping $[\xi,\eta]$ is called the Lie derivative of the multilinear mapping η with respect to the linear mapping ξ.

3. The linear mapping ξ is called a derivation with respect to the multilinear mapping η if we have $[\xi,\eta] = 0$.

Proposition 12
Suppose that ξ is a linear mapping and let η be a multilinear mapping on a unital module over a commutative ring. We have

$$[\xi,\eta](x_1,\dots,x_n) = \xi\eta(x_1,\dots,x_n) - \sum_{m=1}^{n} \eta(x_1,\dots,x_{m-1},\xi x_m,x_{m+1},\dots,x_n)$$

for any elements $(x_1,\dots,x_n)$.

Proposition 13
Suppose that ξ is a linear mapping on a unital module over a commutative ring and let x be an element of the unital module. We have

$$[\xi,x] = \xi x.$$

Proposition 14

Suppose that A is a unital module over a commutative ring R. The mapping

$$\hom_R(A,A) \times \hom_R(A^{\otimes n},A) \to \hom_R(A^{\otimes n},A), \qquad (\xi,\eta) \mapsto [\xi,\eta]$$

is bilinear over the commutative ring R.

Proposition 15

Suppose that A is a unital module over a commutative ring R and let η be a multilinear mapping on the unital module A. The set $\operatorname{der}_\eta A$ of derivations with respect to the multilinear mapping η is a submodule of the endomorphism algebra $\hom_R(A,A)$.

Proposition 16

Suppose that A is a unital module over a commutative ring R and let η be a multilinear mapping on the unital module A. The set

$$\operatorname{der}^n_\eta A = \Big\{ \xi \in \hom_R(A^{\otimes n},A) : \xi \text{ is a multiderivation} \\ \text{with respect to the multilinear mapping } \eta \Big\}$$

is a submodule of the unital module $\hom_R(A^{\otimes n},A)$.

Proposition 17

Suppose that μ is a multilinear mapping on a unital module over a commutative ring and let ξ and η be derivations with respect to the multilinear mapping μ. The commutator $[\xi,\eta]$ is a derivation with respect to the multilinear mapping μ.

Proof. We have

$$\begin{aligned}[\xi,\eta] \circ \mu &= (\xi \circ \eta - \eta \circ \xi) \circ \mu \\ &= \mu \circ \sum_{m=1}^{n} \delta^{\otimes(m-1)} \otimes [\xi,\eta] \otimes \delta^{\otimes(n-m)}.\end{aligned}$$

□

Proposition 18

Suppose that μ is a multilinear mapping on a unital module over a commutative ring. Suppose that ξ is a derivation and let η be a multiderivation with respect to the multilinear mapping μ. The Lie derivative $[\xi,\eta]$ is a multiderivation with respect to the multilinear mapping μ.

Proposition 19

The Lie derivative $[\xi,\eta]$ of a smooth contravariant tensor field η with respect to a smooth vector field ξ on a smooth manfiold is a smooth contravariant tensor field on the smooth manifold.

Proposition 20

Suppose that U is an open submanifold of a smooth manifold X. The following diagram commutes.

$$\begin{array}{ccc} \operatorname{der}^n C^\infty(X) & \longrightarrow & \operatorname{der}^n C^\infty(X) \\ \downarrow & & \downarrow \\ \operatorname{der}^n C^\infty(U) & \longrightarrow & \operatorname{der}^n C^\infty(U) \end{array}, \qquad \eta \mapsto [\xi, \eta].$$

Proposition 21

Suppose that A is a unital module over a commutative ring and let μ be a bilinear mapping on the unital module A. Suppose that ξ is a derivation with respect to the bilinear mapping μ and let η and ζ be multilinear mappings on the unital module A. We have

$$[\xi, \eta \otimes_\mu \zeta] = [\xi, \eta] \otimes_\mu \zeta + \eta \otimes_\mu [\xi, \zeta].$$

Proof. We have

$$\begin{aligned} [\xi, \eta \otimes_\mu \zeta] &= \xi \circ \mu \circ (\eta \otimes \zeta) - \mu \circ \left(\left(\eta \circ \sum_{k=1}^{m} \delta^{\otimes(k-1)} \otimes \xi \otimes \delta^{\otimes(m-k)} \right) \otimes \zeta \right) \\ &\qquad - \mu \circ \left(\eta \otimes \left(\zeta \circ \sum_{k=1}^{n} \delta^{\otimes(k-1)} \otimes \xi \otimes \delta^{\otimes(n-k)} \right) \right) \\ &= \left(\xi \circ \eta - \eta \circ \sum_{k=1}^{m} \delta^{\otimes(k-1)} \otimes \xi \otimes \delta^{\otimes(m-k)} \right) \otimes_\mu \zeta \\ &\quad + \eta \otimes_\mu \left(\xi \circ \zeta - \zeta \circ \sum_{k=1}^{n} \delta^{\otimes(k-1)} \otimes \xi \otimes \delta^{\otimes(n-k)} \right) \\ &= [\xi, \eta] \otimes_\mu \zeta + \eta \otimes_\mu [\xi, \zeta]. \end{aligned}$$

□

Proposition 22

Suppose that ξ is a smooth vector field and let η and ζ be smooth contravariant tensor fields on a smooth manifold. We have

$$[\xi, \eta \otimes \zeta] = [\xi, \eta] \otimes \zeta + \eta \otimes [\xi, \zeta].$$

Proposition 23

Suppose that M is a unital module and let A be a commutative algebra with identity over a commutative ring R. The unital module $\hom_R(M, A)$ is a unital module over the commutative algebra A.

Proposition 24

Suppose that M is a unital module and let A be an algebra over a commutative ring R. The unital module $\hom_R(M, A)$ is a bimodule over the algebra A.

PROPOSITION 25

Suppose that ξ is a derivation and let η be a multilinear mapping on an algebra over a commutative ring with identity. Suppose that x is an element of the algebra. We have

$$[\xi, x\eta] = (\xi x)\eta + x[\xi, \eta].$$

References

[1] Camille Laurent-Gengoux, Anne Pichereau, and Pol Vanhaecke. *Poisson structures*. Die Grundlehren der mathematischen Wissenschaften. Springer, 2013.