Argument Shift Method on Algebra $U\mathfrak{gl}_d$

Yasushi Ikeda

Cracow University of Technology

February 20, 2026

Motivation 1 – Integrable Systems

Let (M, π) be a Poisson manifold.

Integrable Systems

A central objective is to construct a sufficiently large Poisson-commutative family in $C^\infty M$.

A classical method to produce such families is the argument-shift construction of Mishchenko and Fomenko.

Theorem (A. Mishchenko and A. Fomenko, 1978)

Let ξ be a vector field on M with $\mathcal{L}_\xi^2 \pi = 0$. We have

$$\{\xi^m x, \xi^n y\} = 0, \qquad \forall m, \forall n$$

for any Casimir functions x and $y \in C^\infty M$.

Motivation 2 – Deformation Quantization

A deformation quantization of the Poisson manifold M is an $\mathbb{R}[[\nu]]$-bilinear associative product on $(C^\infty M)[[\nu]]$ of the form

$$x \star y = xy + \frac{\nu\{x, y\}}{2} + \sum_{n=2}^{\infty} \nu^n B_n(x, y), \qquad x, y \in C^\infty M,$$

where each B_n is a bidifferential operator. The first-order term recovers the Dirac correspondence rule

$$\{x, y\} = \left[\frac{x \star y - y \star x}{\nu}\right]_{\nu=0}.$$

Motivation 2 – Deformation Quantization

Quantum Integrable Systems

A central objective is to construct a sufficiently large $\star$-commutative family in $(C^\infty M)[[\nu]]$.

Motivation

A quantized argument-shift construction (when it exists) is expected to yield commutation relations of the form

$$\left(\hat{\xi}^m x\right) \star \left(\hat{\xi}^n y\right) = \left(\hat{\xi}^n y\right) \star \left(\hat{\xi}^m x\right), \qquad \forall m, \forall n$$

for any $\star$-central elements x and $y \in (C^\infty M)[[\nu]]$.

Quantum Partial Derivatives

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra.

1. The dual space $\mathfrak{g}^*$ is a Poisson manifold.

$$\pi(x) = \sum_{i<j} \sum_{k=1}^{d} C_{ij}^{k} x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

2. Let ξ be an element of the Poisson manifold $\mathfrak{g}^*$. The vector field $\overline{\partial}_\xi$ in the direction ξ is given by

$$\overline{\partial}_\xi = \sum_{i=1}^{d} \xi(x_i) \frac{\partial}{\partial x_i} \equiv \sum_{i=1}^{d} \xi_i \overline{\partial}^i.$$

Quantum Partial Derivatives

Consider a deformation quantization.

- The universal enveloping algebra $U\mathfrak{g}$ describes a deformation quantization of the Poisson manifold $\mathfrak{g}^*$ at $\nu = 1$.
- The symmetric algebra $S\mathfrak{g}$ is the algebra of polynomial functions on the Poisson manifold $\mathfrak{g}^*$ and the associated graded algebra of the universal enveloping algebra $U\mathfrak{g}$.
- It suffices to construct suitable quantum partial derivatives ∂^i in order to quantize the classical shift operator $\overline{\partial}_\xi$.
- Gurevich, Pyatov, and Saponov introduced such quantum partial derivatives ∂^i_j for $\mathfrak{g} = \mathfrak{gl}_d$.

Quantum Partial Derivatives

We restrict to the case $\mathfrak{g} = \mathfrak{gl}_d$. Let e^i_j be the standard basis elements of $\mathfrak{gl}_d$ and set

$$e = \begin{pmatrix} e^1_1 & \dots & e^1_d \\ \hdotsfor{3} \\ e^d_1 & \dots & e^d_d \end{pmatrix}.$$

They satisfy the commutation relations

$$[e^{i_1}_{j_1}, e^{i_2}_{j_2}] = \delta^{i_1}_{j_2} e^{i_2}_{j_1} - \delta^{i_2}_{j_1} e^{i_1}_{j_2}.$$

We define an operator on the symmetric algebra $S\mathfrak{gl}_d$

$$\overline{\partial} x = \begin{pmatrix} \overline{\partial}^1_1 x & \dots & \overline{\partial}^1_d x \\ \hdotsfor{3} \\ \overline{\partial}^d_1 x & \dots & \overline{\partial}^d_d x \end{pmatrix}, \qquad \overline{\partial}^i_j = \frac{\partial}{\partial e^j_i}.$$

Quantum Partial Derivatives

Remark

The partial derivatives

$$S\mathfrak{gl}_d \to S\mathfrak{gl}_d, \qquad x \mapsto \bar{\partial}^i_j x$$

are the unique linear mappings satisfying the following.

1. $\bar{\partial}\nu = 0$ for any scalar ν.
2. $\bar{\partial}\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (Leibniz rule)

$$\bar{\partial}(xy) = (\bar{\partial}x)y + x(\bar{\partial}y)$$

for any elements x and y of $S\mathfrak{gl}_d$.

Quantum Partial Derivatives

- There are no such derivations of $U\mathfrak{gl}_d$ since $U\mathfrak{gl}_d$ is noncommutative: the Leibniz rule implies
$$\partial\left(e^{i_1}_{j_1}e^{i_2}_{j_2}-e^{i_2}_{j_2}e^{i_1}_{j_1}\right)=0,$$
while $\partial\left(\delta^{i_1}_{j_2}e^{i_2}_{j_1}-\delta^{i_2}_{j_1}e^{i_1}_{j_2}\right)=\delta^{i_1}_{j_2}E^{i_2}_{j_1}-\delta^{i_2}_{j_1}E^{i_1}_{j_2}\neq 0.$
- One can modify the Leibniz rule and introduce well-defined quantum partial derivatives of $U\mathfrak{gl}_d$.

Quantum Partial Derivatives

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quantum partial derivatives

$$U\mathfrak{gl}_d \to U\mathfrak{gl}_d, \qquad x \mapsto \partial^i_j x$$

are the unique linear mappings satisfying the following.

1. $\partial \nu = 0$ for any scalar ν.
2. $\partial\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (quantum Leibniz rule)

$$\partial(xy) = (\partial x)y + x(\partial y) + \boxed{(\partial x)(\partial y)}$$

for any elements x and y of $U\mathfrak{gl}_d$.

Quantum Partial Derivatives

Remark

The linear operator

$$U\mathfrak{gl}_d \to M_d(U\mathfrak{gl}_d), \qquad x \mapsto \mathrm{diag}(x, \dots, x) + \partial x \equiv Dx$$

is a unital algebra homomorphism by the quantum Leibniz rule.

- Equivalently,

  $$D = (\mathrm{id} \otimes \rho) \circ \Delta,$$

  where Δ denotes the comultiplication on $U\mathfrak{gl}_d$ and $\rho\colon U\mathfrak{gl}_d \to M_d$ is the extension of the standard representation of $\mathfrak{gl}_d$ on $\mathbb{C}^d$.
- We may define the quantum partial derivatives ∂^i_j either explicitly or using the homomorphism $D = (\mathrm{id} \otimes \rho) \circ \Delta$.

Main Theorem

Suppose ξ is a numerical matrix and let $D_\xi = \mathrm{tr}(\xi D)$. The following is our main theorem.

Theorem (I. and Sharygin, 2024)

$\bigcup_{n=0}^{\infty}\left\{ D_\xi^n x : x \text{ is central} \right\}$ *is a commuting family.*

- We may assume that ξ is diagonal with diagonal entries $(z_1, \dots, z_d)$ mutually distinct since such matrices with their conjugates form a dense subset of M_d.
- Vinberg and Rybnikov showed that the centralizer of

$$\left\{ e_i^i, \sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j} \right\}_{i=1}^{d}$$

is a commutative subalgebra of $U\mathfrak{gl}_d$.

Main Theorem

- The proof is carried out by showing that the quantum shift $\partial_\xi^n x$ commutes with these elements by induction on n.
- It reduces to proving

$$\left[\sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j}, D_\xi x\right] = 0, \quad \left[\left[\operatorname{ad} \sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j}, D_\xi\right], D_\xi\right] = 0.$$

- The second condition can be verified by direct computation.
- The first condition reduces to computing the first-order quantum shift of an arbitrary central element.

Formula

We compute the quantum shifts $D_\xi^n x$ for an arbitrary central element x. The center C of $U\mathfrak{gl}_d$ is a free commutative algebra generated by

$$\operatorname{tr} e, \qquad \dots, \qquad \operatorname{tr} e^d.$$

They are the Gelfand invariants. It is necessary and even sufficient to compute the quantum partial derivatives $D(e^n)^i_j$.

Formula

I obtained the following formula for the quantum partial derivatives. Define $f_{\pm}^{(n)}(x) = \sum_{m=0}^{n} \frac{1 \pm (-1)^{n-m}}{2} \binom{n-1}{m} x^m$.

Theorem (I, 2022)

We have

$$D(e^n)^i_j = \sum_{m=0}^{n} (f_+^{(n-m)}(e)(e^m)^i_j + f_-^{(n-m)}(e)_j(e^m)^i) = \sum_{m=0}^{n} ((e^m)^i_j f_+^{(n-m)}(e) + (e^m)_j f_-^{(n-m)}(e)^i)$$

for any nonnegative integer n.

Formula

Notation

We denote the i-th row and j-th column of a matrix A by

$$A^i = \begin{pmatrix} A^i_1 & \ldots & A^i_d \end{pmatrix}, \qquad A_j = \begin{pmatrix} A^1_j \\ \vdots \\ A^d_j \end{pmatrix}.$$

This formula is used to establish the base case of the induction.

Formula

We assume the following form

$$D(e^n)^i_j = \sum_{m=0}^{n} \left(g^{(n)}_m(e)(e^m)^i_j + h^{(n)}_m(e)_j(e^m)^i \right),$$

where $g^{(n)}_m$ and $h^{(n)}_m$ are polynomials. By the quantum Leibniz rule and the commutation relations

$$\left[(e^m)^i, e^k_j \right] = (e^m)^k \delta^i_j - \delta^k (e^m)^i_j,$$

We obtained the initial condition

$$g^{(0)}_0(x) = 1, \qquad h^{(0)}_0(x) = 0$$

Formula

and the recursive relation

$$
\begin{aligned}
g_0^{(n+1)}(x) &= \sum_{m=0}^{n} h_m^{(n)}(x)x^m, \\
g_m^{(n+1)}(x) &= g_{m-1}^{(n)}(x), & 0 < m \leq n+1, \\
h_m^{(n+1)}(x) &= g_m^{(n)}(x) + h_m^{(n)}(x)x, & 0 \leq m < n+1, \\
h_{n+1}^{(n+1)}(x) &= 0.
\end{aligned}
$$

Its solution is

$$
g_m^{(n)}(x) = f_+^{(n-m)}(x), \qquad h_m^{(n)}(x) = f_-^{(n-m)}(x).
$$

Generators

Our theorem yields an increasing sequence of commutative subalgebras

$$C_\xi^{(0)} = C, \qquad C_\xi^{(n)} = C_\xi^{(n-1)}\left[D_\xi^n C\right].$$

Using the formula above we obtain

$$C_\xi^{(1)} = C_\xi^{(0)}\left[\operatorname{tr}\left(\xi e^n\right) : n = 1, 2, \ldots\right],$$
$$C_\xi^{(2)} = C_\xi^{(1)}\left[\tau_\xi \begin{pmatrix} 0 & P_n^\top \\ P_m & 0 \end{pmatrix} : |m - n| \leq 1\right].$$

Generators

Here let $P_n^{(m)}$ $(m, n \in \mathbb{Z}_{\geq 0})$ be the matrices of binomial coefficients determined by the formal identity

$$x^m f_-^{(n-j+1)}(x) = \sum_{i=1}^{m+n} \left(P_n^{(m)}\right)_j^i x^{i-1},$$

where x is a formal variable. We also set $P_n = P_n^{(0)}$. For any matrix A define

$$\tau_\xi(A) = \sum_{i,j} A_j^i \operatorname{tr}\left(\xi e^{i-1} \xi e^{j-1}\right).$$

Generators

To describe $C_\xi^{(2)}$ we need the following lemma.

Lemma

We have

$$\sigma\begin{pmatrix} 0 & P_m^\top \\ P_{m+2n} & 0 \end{pmatrix} = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right) P_{m+k}^{(m+k)},$$

$$\sigma\begin{pmatrix} 0 & P_m^\top \\ P_{m+2n+1} & 0 \end{pmatrix} = \sum_{k=0}^{n}\binom{2n-k}{k}\left(P_{m+k+1}^{(m+k)} + P_{m+k}^{(m+k+1)}\right).$$

for any nonnegative integers m and n.

Generators

Define the mapping σ on $n \times n$ matrices x by

$$\sigma(x) = \begin{pmatrix} x_1^1 & 0 & \cdots & 0 \\ x_1^2 + x_2^1 & x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n + x_n^1 & x_2^n + x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

The lemma is equivalent to the following identities.

Generators

For any nonnegative integers n_1, n_2, n_3, we have

$$\binom{2n_1+n_2+2n_3+1}{2n_3}+\binom{n_2+2n_3}{2n_3} = \sum_{n_4=0}^{n_3}\left(\binom{n_1+n_2+n_3+n_4+1}{2n_4}+\binom{n_1+n_2+n_3+n_4}{2n_4}\right)\binom{n_1+n_3-n_4}{2(n_3-n_4)}.$$

Generators

Moreover,

$$\binom{2n_1+n_2+2n_3+2}{2n_3}+\binom{n_2+2n_3}{2n_3} = \sum_{n_4=0}^{n_3}\binom{n_1+n_2+n_3+n_4+1}{2n_4}\left(\binom{n_1+n_3-n_4+1}{2(n_3-n_4)}+\binom{n_1+n_3-n_4}{2(n_3-n_4)}\right).$$

Generators

For any nonnegative integers m and n, we have

$$f_-^{(m+2n+1)}(x) + f_-^{(m+1)}(x)x^{2n} = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right) f_-^{(m+k+1)}(x)x^k$$

and

$$f_-^{(m+2n+2)}(x) + f_-^{(m+1)}(x)x^{2n+1} = \sum_{k=0}^{n}\binom{2n-k}{k}\left(f_-^{(m+k+2)}(x)x^k + f_-^{(m+k+1)}(x)x^{k+1}\right).$$

Generators

These identities reduce to the following binomial relations.

1. For $\varepsilon = 0, 1$,

$$\binom{x+y+n}{2n+\varepsilon} + \binom{x-y+n}{2n+\varepsilon} = \sum_{m=0}^{n} \binom{x+m}{2m+\varepsilon} \left(\binom{y+n-m}{2(n-m)} + \binom{y-1+n-m}{2(n-m)} \right).$$

2. $$\sum_{m=0}^{n} \binom{x-m}{m}\binom{y+m}{n-m} = \sum_{m=0}^{n} \binom{x+y-m}{m}\binom{m}{n-m}.$$

3. $$\binom{x}{n} = \sum_{m=0}^{n} \binom{x-m}{m}\binom{m}{n-m} + \sum_{m=0}^{n-1} \binom{x-1-m}{m}\binom{m}{n-1-m}.$$

Generators

The generators are $\mathrm{tr}(\xi e)$, $\mathrm{tr}(\xi e^2)$, $\ldots$ and

$$\begin{aligned}
&\mathrm{tr}(\xi^2 e),\\
&\mathrm{tr}(2\xi^2 e^2 + \xi e \xi e),\\
&\mathrm{tr}(\xi^2 e^3 + \xi e \xi e^2),\\
&\mathrm{tr}(2\xi^2 e^4 + 2\xi e \xi e^3 + \xi e^2 \xi e^2 + \xi^2 e^2),\\
&\mathrm{tr}(\xi^2 e^5 + \xi e \xi e^4 + \xi e^2 \xi e^3 + \xi^2 e^3),\\
&\mathrm{tr}(2\xi^2 e^6 + 2\xi e \xi e^5 + 2\xi e^2 \xi e^4 + \xi e^3 \xi e^3 + 4\xi^2 e^4 + \xi e \xi e^3),\\
&\mathrm{tr}(\xi^2 e^7 + \xi e \xi e^6 + \xi e^2 \xi e^5 + \xi e^3 \xi e^4 + 3\xi^2 e^5 + \xi e \xi e^4), \ldots.
\end{aligned}$$

They are mutually commutative.