Real Analysis

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Contents

1 The Lebesgue Spaces

Suppose that (X, μ) is a measure space.

Remark 1.1.

$$\mathcal{L}^\infty(\mu) = \left\{ f : \|f\|_\infty = \min\{ m : |f| \leq m \} = \min_{\mu(N)=0} \sup_{x \in X \setminus N} |f(x)| < \infty \right\}$$

is a *-subalgebra of the *-algebra of measurable functions.

Remark 1.2.

$$\mathcal{L}^p(\mu) = \left\{ f : \|f\|_p = (\int |f|^p \, d\mu)^{1/p} < \infty \right\}$$

is a subspace of the *-algebra of measurable functions for $1 \leq \forall p < \infty$.

Definition 1.1. We denote the *-algebra of equivalence classes of elements of the *-algebra of measurable functions by $M(\mu)$.

Remark 1.3.

1. $M(\mu)$ is a *-algebra.
2. $L^\infty(\mu)$ is a *-subalgebra of $M(\mu)$.
3. $L^\infty(\mu)$ is a C^*-algebra.
4. $L^\infty(\mu) = L^1(\mu)^*$ provided that (X, μ) is σ-finite.

5. $L^p(\mu)$ is a subspace of $M(\mu)$ for $1 \leq \forall p \leq \infty$.
6. $L^p(\mu)$ is a complex Banach space for $1 \leq \forall p \leq \infty$.
7. $L^1(\mu)$ is a closed subspace of $L^\infty(\mu)^*$.
8. $L^2(\mu)$ is a Hilbert space.

Proposition 1.1. *Suppose that x and y are non-negative real numbers.*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

for real Hölder conjugates $\forall p$ and $\forall q$.

Suppose that f, g, f_1, f_2, $\ldots$ are measurable functions.

Remark 1.4.

$$\int |f|^p \, d\mu < \infty \Leftrightarrow \int |\mathrm{Re}\, f|^p \, d\mu, \int |\mathrm{Im}\, f|^p \, d\mu < \infty$$

for $0 < \forall p < \infty$.

Theorem 1.1 (Hölder's Inequality)**.**

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

for extended real Hölder conjugates $\forall p$ and $\forall q$.

Remark 1.5.

$$\|fg\|_1 = \|f\|_1 \|g\|_\infty \Leftrightarrow |fg| = |f| \|g\|_\infty$$

provided that f is integrable and g is essentially bounded.

Corollary 1.1 (Schwarz's Inequality)**.**

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2.$$

Theorem 1.2 (Minkowski's Inequality)**.**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

for $1 \leq \forall p \leq \infty$.

Remark 1.6. 1. $(f_n)_{n=1}^\infty$ converges to f in measure μ if

$$\|f\|_p, \|f_1\|_p, \|f_2\|_p, \cdots < \infty, \qquad \overset{p}{\lim_{n\to\infty}} f_n = f$$

for $1 \leq \forall p \leq \infty$.

2. $\lim^p_{n\to\infty} f_n = f$ if $(f_n)_{n=1}^\infty$ converges to f in measure μ and

$$\|f\|_p, \|f_1\|_p, \|f_2\|_p, \cdots < \infty, \qquad \lim_{n\to\infty} \|f_n\|_p = \|f\|_p$$

for $1 \leq \forall p < \infty$.

3. $\lim^p_{n\to\infty} f_n = f$ if $\lim_{n\to\infty} f_n = f$ and

$$\|f\|_p, \|f_1\|_p, \|f_2\|_p, \cdots < \infty, \qquad \lim_{n\to\infty} \|f_n\|_p = \|f\|_p$$

for $1 \leq \forall p < \infty$.

2 Convergence a.e. and Convergence in Measure

Suppose that $f,\ f',\ f_1,\ f_2,\ \dots$ are measurable functions on a measure space (X, μ).

Remark 2.1. $\lim_{n\to\infty} f_n = f$ if and only if

$$\mu\left(\limsup_{n\to\infty}\left\{\, x : |f_n(x) - f(x)| \geq \frac{1}{m} \,\right\}\right) = 0$$

for $\forall m = 1, 2, \dots$

Theorem 2.1. *1. $\lim_{n\to\infty} f_n = f$ if there exists a sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$ and*

$$\sum_{n=1}^{\infty} \mu(\{\, x : |f_n(x) - f(x)| \geq \varepsilon_n \,\}) < \infty.$$

2. $\sum_{n=1}^{\infty} f_n$ converges absolutely if there exists a sequence $(\eta_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \eta_n < \infty$ and

$$\sum_{n=1}^{\infty} \mu(\{\, x : |f_n(x)| \geq \eta_n \,\}) < \infty.$$

Theorem 2.2 (Lebesgue's Dominated Convergence Theorem)**.** *If $\lim_{n\to\infty} f_n = f$ and there exists a non-negative integrable function g such that*

$$\sum_{n=1}^{\infty} \mu(\{\, x : g(x) < |f_n(x)| \,\}) = 0$$

then f is integrable and $\lim^1_{n\to\infty} f_n = f$.

Theorem 2.3 (Dominated Convergence Theorem under Convergence in Measure)**.** *If $(f_n)_{n=1}^{\infty}$ converges to f in measure μ and there exists a non-negative integrable function g such that*

$$\sum_{n=1}^{\infty} \mu(\{\, x : g(x) < |f_n(x)| \,\}) = 0$$

then f is integrable and $\lim^1_{n\to\infty} f_n = f$.

Definition 2.1. $(f_n)_{n=1}^{\infty}$ is said to converge almost uniformly to f if there exists a measurable set N_η such that $\mu(N_\eta) < \eta$ and

$$\lim_{n\to\infty} \sup_{x \in X \setminus N_\eta} |f_n(x) - f(x)| = 0$$

for $\forall \eta > 0$.

Remark 2.2. If $(f_n)_{n=1}^{\infty}$ converges almost uniformly to f then $\lim_{n\to\infty} f_n = f$.

Theorem 2.4 (D. E. Egoroff). *If $\lim_{n\to\infty} f_n = f$ then $(f_n)_{n=1}^{\infty}$ converges almost uniformly to f provided that $\mu(X) < \infty$.*

Definition 2.2. $(f_n)_{n=1}^{\infty}$ is said to converge to f in measure μ if

$$\lim_{n\to\infty} \mu(\{\, x : |f_n(x) - f(x)| \geq \varepsilon \,\}) = 0$$

for $\forall \varepsilon > 0$.

Remark 2.3.

1. If $(f_n)_{n=1}^{\infty}$ converges to f and f' in measure μ then $f = f'$.
2. $(f_n)_{n=1}^{\infty}$ converges to f in measure μ if and only if there exists N_m such that
$$\sup_{n\geq N_m} \mu\left(\left\{\, x : |f_n(x) - f(x)| \geq \frac{1}{m} \,\right\}\right) \leq \frac{1}{m}$$
for $\forall m = 1, 2, \ldots$

Theorem 2.5 (F. Riesz). *If $(f_n)_{n=1}^{\infty}$ converges to f in measure μ then there exists a subsequence of $(f_n)_{n=1}^{\infty}$ converging to f.*

Theorem 2.6 (H. Lebesgue). *If $\lim_{n\to\infty} f_n = f$ then $(f_n)_{n=1}^{\infty}$ converges to f in measure μ provided that $\mu(X) < \infty$.*

Definition 2.3. $(f_n)_{n=1}^{\infty}$ is called a Cauchy sequence with respect to convergence in measure μ if

$$\lim_{m,n\to\infty} \mu(\{\, x : |f_m(x) - f_n(x)| \geq \varepsilon \,\}) = 0$$

for $\forall \varepsilon > 0$.

Remark 2.4. The following are equivalent.

1. $(f_n)_{n=1}^{\infty}$ converges in measure μ.
2. $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence with respect to convergence in measure μ.
3. There exists N_δ such that
$$\sup_{m,n\geq N_\delta} \mu(\{\, x : |f_m(x) - f_n(x)| \geq \delta \,\}) \leq \delta$$
for $\forall \delta > 0$.

3 Complex Measures

Definition 3.1. A real-valued (resp. complex-valued) function on a σ-algebra is called a signed (resp. complex) measure if it is countably additive.

Remark 3.1. Suppose that (X, μ) is a complex measure space.

$$\lim_{n\to\infty} \mu(S_n) = \mu\Big(\lim_{n\to\infty} S_n\Big)$$

for a monotone sequence $\forall (S_n)_{n=1}^{\infty}$ of measurable sets.

Example 3.1. Suppose that f is an integrable function on a measure space (X, μ). $S \mapsto \int_S f \, d\mu$ is a complex measure on X. $S \mapsto \int_S |f| \, d\mu$ is the total variation.

Remark 3.2.

$$|\mathrm{Re}\,\mu|, |\mathrm{Im}\,\mu| \le |\mu| \le |\mathrm{Re}\,\mu| + |\mathrm{Im}\,\mu|$$

for a complex measure $\forall \mu$.

Remark 3.3.

$$|\mu + \nu| \le |\mu| + |\nu|, \qquad ||\mu| - |\nu|| \le |\mu - \nu|$$

for complex measures $\forall \mu$ and $\forall \nu$ on a measurable space.

Proposition 3.1. *Suppose that g is an integrable (resp. a non-negative measurable) function on a measure space (X, μ).*

$$\int f \, d \int_{\cdot} g \, d\mu = \int f g \, d\mu$$

for a measurable function $\forall f$.

Definition 3.2. Suppose that X is a measurable space. Two complex measures μ_1 and μ_2 are said to be mutually singular and we write $\mu_1 \perp \mu_2$ if there exist two measurable sets X_1 and X_2 such that X is a disjoint union of X_1 and X_2 and $|\mu_1|(X_2) = |\mu_2|(X_1) = 0$.

Theorem 3.1 (Jordan Decomposition)**.** *Suppose that (X, μ) is a signed measure space. There exists a unique pair (μ^+, μ^-) of finite measures such that $\mu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$.*

Remark 3.4.

$$|\mu| = \mu^+ + \mu^-$$

for a signed measure $\forall \mu$.

Definition 3.3. Suppose that (X, μ) is a signed measure space. A measurable set S is called a positive (resp. negative) set if (S, μ) (resp. $(S, -\mu)$) is a measure space.

Remark 3.5. A union of countable positive (resp. negative) sets is a positive (resp. negative) set.

Theorem 3.2 (Hahn Decomposition)**.** *A Hahn decomposition of a signed measure space exists.*

Theorem 3.3 (Lebesgue Decomposition)**.** *Suppose that (X, μ) is a σ-finite measure space and that ν is a complex (resp. finite) measure on X. There exists a unique pair (ν_1, ν_2) of complex (resp. finite) measures satisfying the following.*

1. $\nu = \nu_1 + \nu_2$.
2. ν_1 *is absolutely continuous with respect to* μ.
3. ν_2 *and* μ *are mutually singular.*

Theorem 3.4. *The set of complex measures on a measurable space is a complex Banach space.*

Theorem 3.5 (Radon-Nikodym)**.** *$L^1(X, \mu)$ is the Banach space of complex measures on X absolutely continuous with respect to μ for a σ-finite measure space $\forall(X, \mu)$.*

Proposition 3.2. *Suppose that f is an integrable function on a probability space (X, μ).*

$$\mu\left(f^{-1}\left(\overline{\left\{\int_S f\, d\mu : \mu(S) > 0\right\}}\right)\right) = 1.$$

Proposition 3.3. *The Radon-Nikodym derivative of a complex measure with respect to the total variation is of modulus one.*

Proposition 3.4. *Suppose that X is a measurable space. A complex measure ν on X is absolutely continuous with respect to a measure μ on X if and only if $\inf_{\delta>0} \sup_{\mu(S)<\delta} |\nu(S)| = 0$.*

4 Integration on a Locally Compact Hausdorff Space

Definition 4.1. A measure on a locally compact Hausdorff space that is finite on the set of compact sets is called a Borel measure.

Definition 4.2. A Borel measure μ on a locally compact Hausdorff space is called a Radon measure if it satisfies the following.

1. μ is inner regular for any open set.
2. μ is outer regular.

Remark 4.1. A Radon measure on a locally compact Hausdorff space is inner regular for any Borel set of finite measure.

Theorem 4.1 (Monotone Convergence Theorem)**.** *Suppose that μ is a Radon measure on a locally compact Hausdorff space and that $(f_i)_i$ is an increasing net of non-negative extended real-valued lower semi-continuous functions on the space.*

$$\int \sup_i f_i\, d\mu = \sup_i \int f_i\, d\mu.$$

Proposition 4.1. *If μ is a Radon measure on a locally compact Hausdorff space then $\int_{\cdot} f\, d\mu$ is a Radon measure on the space for a strictly positive continuous function $\forall f$ on the space.*

Definition 4.3. A signed measure μ on a locally compact Hausdorff space is called a signed Radon measure if μ^+ and μ^- are Radon measures.

Definition 4.4. A complex measure μ on a locally compact Hausdorff space is called a complex Radon measure if $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$ are signed Radon measures.

Remark 4.2. The total variation of a complex Radon measure on a locally compact Hausdorff space is a Radon measure.

Theorem 4.2. *A Borel measure on a locally compact Hausdorff space is a Radon measure if any open set is σ-compact.*

Proposition 4.2. *A complex measure on a locally compact Hausdorff space that is absolutely continuous with respect to a finite Radon measure is a complex Radon measure.*

Theorem 4.3. *Suppose that f is an integrable function with respect to a finite Radon measure on a compact Hausdorff space. There exists a sequence $(f_n)_{n=1}^{\infty}$ of continuous functions such that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.*

Remark 4.3. There exists the largest open null set $X \setminus \operatorname{supp} \mu$ for a measure $\forall \mu$ on a second countable topological space X.

5 Fourier Analysis

Theorem 5.1. *The convolution of rapidly decreasing functions is a rapidly decreasing function and the Fourier transform of the convolution is the product of the Fourier transforms.*

Proposition 5.1. *There exists a non-zero real-valued odd compactly supported smooth function on $\mathbb{R}^d$ if d is positive.*

Theorem 5.2. *There exists a non-zero non-negative rapidly decreasing function on $\mathbb{R}^d$ such that the Fourier transform is non-negative and compactly supported.*

Proof. We may assume that d is positive. There exists a rapidly decreasing function f such that the Fourier transform of f is non-zero, real-valued, odd, and compactly supported. $|f * \bar{f}|^2$ is a rapidly decreasing function satisfying the desired properties. □

Theorem 5.3. *There exists a sequence $(f_n)_{n=1}^{\infty}$ of non-zero non-negative rapidly decreasing functions on $\mathbb{R}^d$ satisfying the following.*

1. $$\int f_n(x)\, dx = 1$$

   and the Fourier transform of f_n is non-negative and compactly supported for $\forall n$.

2.

$$\lim_{n\to\infty}\int f_n(x)g(x)\,dx = g(0)$$

for a bounded continuous function $\forall g$ on $\mathbb{R}^d$.

Proof. We may assume that d is positive. There exists a non-zero non-negative rapidly decreasing function f on $\mathbb{R}^d$ such that the Fourier transform is non-negative and compactly supported. We may assume that

$$\int f(x)\,dx = 1.$$

We define $f_n(x) = nf(n^{1/d}x)$. $(f_n)_{n=1}^{\infty}$ is a sequence of functions satisfying the desired properties. □

References

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