Quantum partial derivatives and argument shift method

Yasushi Ikeda

Moscow State University

November 10, 2024

Outline

Algebra

Definition (Algebra)

An *algebra* is a vector space A with an associative bilinear mapping

$$A^2 \to A, \qquad (x, y) \mapsto xy.$$

- The set $M(d, \mathbb{C})$ of d by d complex matrices is an complex algebra.
- Suppose that H is a Hilbert space. The set $B(H)$ of bounded linear operators on the Hilbert space H is a complex algebra.

Commutator and Lie algebra

Remark (Commutator)

The alternating mapping

$$A^2 \to A, \qquad (x, y) \mapsto [x, y] = xy - yx$$

satisfies the Jacobi identity.

Definition (Lie algebra)

A *Lie algebra* is a vector space g with an alternating mapping

$$g^2 \to g, \qquad (x, y) \mapsto [x, y]$$

satisfying the Jacobi identity.

General linear Lie algebra

1. Any algebra is a Lie algebra.
2. Any Lie algebra is isomorphic to a Lie subalgebra of some algebra by the Poincaré–Birkhoff–Witt theorem.

Remark

The notion of Lie algebras is an abstraction of commutators.

Definition (General linear Lie algebra)

The general linear Lie algebra $gl(d, \mathbb{C}) = M(d, \mathbb{C})$.

Ado's theorem

Theorem (Ado)

Any finite dimensional complex Lie algebra g is isomorphic to a Lie subalgebra of the general linear Lie algebra $gl(d, \mathbb{C})$ for some d.

Suppose that $(e_1, \ldots, e_d)$ is a basis of a finite dimensional complex Lie algebra g.

Symmetric algebra

Definition (Symmetric algebra)

The symmetric algebra Sg is an algebra generated by the elements $(e_1, \ldots, e_d)$ subject to the defining relations

$$e_m e_n = e_n e_m, \qquad m, n = 1, \ldots, d.$$

Remark

The symmetric algebra Sg is nothing but the polynomial algebra $\mathbb{C}[e_1, \ldots, e_d]$.

Universal enveloping algebra

Definition (Universal enveloping algebra)

The universal enveloping algebra $U\mathfrak{g}$ is an algebra generated by the elements $(e_1, \ldots, e_d)$ subject to the defining relations

$$e_m e_n = e_n e_m + [e_m, e_n], \qquad m, n = 1, \ldots, d.$$

Remark

The universal enveloping algebra $U\mathfrak{g}$ is a quantization of the symmetric algebra $S\mathfrak{g}$.

Lie–Poisson bracket

Theorem (Lie–Poisson bracket)

There exists a unique Poisson bracket on the symmetric algebra Sg extending the Lie bracket on the Lie algebra g.

$$\begin{array}{ccc} Sg \times Sg & \xrightarrow{\textit{Poisson bracket}} & Sg \\ \uparrow & & \uparrow \\ g \times g & \xrightarrow{\textit{Lie bracket}} & g \end{array}$$

Remark

We have $\operatorname{gr} Ug = Sg$.

Classical argument shift method

Suppose that ξ is an element of the dual space g^* and let

$$\overline{\partial}_\xi = \xi(e_1)\frac{\partial}{\partial e_1} + \cdots + \xi(e_d)\frac{\partial}{\partial e_d}$$

be a derivation on the symmetric algebra Sg.

Theorem (Mishchenko and Fomenko, 1978)

The family

$$\bigcup_{n=0}^{\infty}\left\{ \overline{\partial}_\xi^n x : x \text{ is Poisson central} \right\} \tag{1}$$

is Poisson commutative.

Argument shift algebra (classical and quantum)

Definition (Argument shift algebra)

The Poisson commutative subalgebra $\overline{C}_\xi$ generated by the Poisson commutative family (1) is called the argument shift algebra in the direction ξ.

Definition (Quantum argument shift algebra)

A commutative subalgebra C_ξ of the universal enveloping algebra $U\mathfrak{g}$ satisfying $\operatorname{gr} C_\xi = \overline{C}_\xi$ is called a quantum argument shift algebra in the direction ξ.

The motivation for my talk is to quantize not only the algebra $\overline{C}_\xi$ but also the derivation $\overline{\partial}_\xi$.

Generating matrix of general linear Lie algebra

Let $e = \begin{pmatrix} e^1_1 & \dots & e^1_d \\ \dots & \dots & \dots \\ e^d_1 & \dots & e^d_d \end{pmatrix}$ be a matrix satisfying the following.

- The set
$$\left\{ e^i_j : i, j = 1, \dots, d \right\}$$
is a basis of the general linear Lie algebra $gl(d, \mathbb{C})$.
- We have the commutation relations
$$\left[e^{i_1}_{j_1}, e^{i_2}_{j_2} \right] = \delta^{i_1}_{j_2} e^{i_2}_{j_1} - \delta^{i_2}_{j_1} e^{i_1}_{j_2}.$$

Partial derivatives

We define

$$\overline{\partial} x = \begin{pmatrix} \overline{\partial}^1_1 x & \dots & \overline{\partial}^1_d x \\ \dots & \dots & \dots \\ \overline{\partial}^d_1 x & \dots & \overline{\partial}^d_d x \end{pmatrix}, \qquad \overline{\partial}^i_j = \frac{\partial}{\partial e^j_i}$$

for any element x of the symmetric algebra $Sgl(d, \mathbb{C})$.

Differential operator

Remark

The differential operator

$$Sgl(d,\mathbb{C}) \to M(d, Sgl(d,\mathbb{C})), \qquad x \mapsto \overline{\partial} x$$

is a unique linear operator satisfying the following.

1. $\overline{\partial}\nu = 0$ for any scalar ν.
2. $\overline{\partial}\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (Leibniz rule)

   $$\overline{\partial}(xy) = (\overline{\partial} x)y + x(\overline{\partial} y)$$

   for any elements x and y of the symmetric algebra $Sgl(d,\mathbb{C})$.

Quantum differential operator

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quantum differential operator

$$Ugl(d,\mathbb{C}) \to M(d, Ugl(d,\mathbb{C})), \qquad x \mapsto \partial x$$

is a unique linear operator satisfying the following.

1. $\partial \nu = 0$ for any scalar ν.
2. $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (quantum Leibniz rule)

   $$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y)$$

   for any elements x and y of the universal enveloping algebra $Ugl(d,\mathbb{C})$.

Quantum argument shift method

Suppose that ξ is a numerical matrix and let $\partial_\xi = \mathrm{tr}(\xi\partial)$.

Theorem (I. and Sharygin, 2023)

The family

$$\bigcup_{n=0}^{\infty} \left\{ \partial_\xi^n x : x \text{ is central} \right\} \tag{2}$$

is commutative.

Corollary

The subalgebra C_ξ generated by the family (2) is a quantum argument shift algebra in the direction ξ.

Gelfand invariants

Remark (Gelfand invariants)

The center C of the universal enveloping algebra $Ugl(d, \mathbb{C})$ is generated by the elements $\left\{\operatorname{tr} e, \dots, \operatorname{tr} e^d\right\}$.

Quantum differential operator (modified)

Remark

The linear operator

$$Ugl(d,\mathbb{C}) \to M\big(d, Ugl(d,\mathbb{C})\big), \quad x \mapsto \operatorname{diag}(x,\dots,x) + \partial x$$

is an algebraic homomorphism and will be denoted by ∂ from now on. We have the *quantum Leibniz rule*

$$\partial(xy) = (\partial x)(\partial y)$$

for any elements x and y of the universal enveloping algebra $Ugl(d,\mathbb{C})$.

Formula

We define

$$f_{\pm}^{(n)}(x) = \sum_{m=0}^{n} \frac{1 \pm (-1)^{n-m}}{2} \binom{n-1}{m} x^m.$$

Theorem (I, 2022)

We have

$$\begin{aligned}\partial(e^n)^i_j &= \sum_{m=0}^{n} \left(f_+^{(n-m)}(e)(e^m)^i_j + f_-^{(n-m)}(e)_j(e^m)^i\right) \\ &= \sum_{m=0}^{n} \left((e^m)^i_j f_+^{(n-m)}(e) + (e^m)_j f_-^{(n-m)}(e)^i\right).\end{aligned}$$

Formula

We assume the following form

$$\partial(e^n)^i_j = \sum_{m=0}^{n} (g^{(n)}_m(e)(e^m)^i_j + h^{(n)}_m(e)_j(e^m)^i),$$

where $g^{(n)}_m$ and $h^{(n)}_m$ are polynomials.

Formula

We have

$$\begin{aligned}\partial(e^{n+1})^i_j &= \sum_{k=1}^{d} \partial\left((e^n)^i_k e^k_j\right) = \sum_{k=1}^{d} \left(\partial(e^n)^i_k\right)\left(\partial e^k_j\right) \\ &= \sum_{m=0}^{n}\sum_{k=1}^{d} \left(g^{(n)}_m(e)(e^m)^i_k + h^{(n)}_m(e)_k(e^m)^i\right)\left(e^k_j + E^k_j\right)\end{aligned}$$

by the quantum Leibniz rule and the induction hypothesis.

Formula

We have

$$\begin{aligned}\partial(e^{n+1})^i_j &= \sum_{m=0}^{n} h_m^{(n)}(e)e^m\delta^i_j + \sum_{m=1}^{n+1} g_{m-1}^{(n)}(e)(e^m)^i_j \\ &\quad + \sum_{m=0}^{n} \big(g_m^{(n)}(e) + h_m^{(n)}(e)e\big)_j (e^m)^i \\ &= \sum_{m=0}^{n+1} \big(g_m^{(n+1)}(e)(e^m)^i_j + h_m^{(n+1)}(e)_j (e^m)^i\big)\end{aligned}$$

by the commutation relations

$$\big[(e^m)^i, e^k_j\big] = (e^m)^k\delta^i_j - \delta^k(e^m)^i_j.$$

Formula

We obtained the initial condition

$$g_0^{(0)}(x) = 1, \qquad h_0^{(0)}(x) = 0$$

and the recursive relation

$$\begin{aligned}
g_0^{(n+1)}(x) &= \sum_{m=0}^{n} h_m^{(n)}(x) x^m, \\
g_m^{(n+1)}(x) &= g_{m-1}^{(n)}(x), && 0 < m \le n+1, \\
h_m^{(n+1)}(x) &= g_m^{(n)}(x) + h_m^{(n)}(x) x, && 0 \le m < n+1, \\
h_{n+1}^{(n+1)}(x) &= 0.
\end{aligned}$$

Formula

Its solution is

$$g_m^{(n)}(x) = f_+^{(n-m)}(x), \qquad h_m^{(n)}(x) = f_-^{(n-m)}(x).$$

Quantum directional derivative

We adhere to the convention that $\operatorname{tr} e^{-1} = 1$. We have

$$\partial_\xi \prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots \operatorname{tr}\big(\xi \prod_k f_+^{(n_k - m_k)}(e)\big),$$

$$\partial_\xi^2 \prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots \sum_{k_1=0}^{n_1-m_1} \sum_{k_2=0}^{n_2-m_2} \cdots$$

$$\operatorname{tr}\Big(\xi \prod_\ell f_+^{(k_\ell)}(e) \partial \operatorname{tr}\big(\xi \prod_\ell f_+^{(n_\ell - m_\ell - k_\ell)}(e)\big)\Big).$$

Generators

We define

$$C_\xi^{(0)} = C, \qquad C_\xi^{(n)} = C_\xi^{(n-1)}\left[\partial_\xi^n C\right].$$

We have

$$\begin{aligned} C_\xi^{(1)} &= C_\xi^{(0)}\left[\operatorname{tr}\left(\xi e^n\right) : n = 1, 2, \dots\right], \\ C_\xi^{(2)} &= C_\xi^{(1)}\left[(\tau_\xi \circ \sigma)\begin{pmatrix} 0 & P_n^T \\ P_m & 0 \end{pmatrix} : m, n = 0, 1, 2, \dots\right] \\ &= C_\xi^{(1)}\left[(\tau_\xi \circ \sigma)\begin{pmatrix} 0 & P_n^T \\ P_m & 0 \end{pmatrix} : |m - n| \le 1\right]. \end{aligned}$$

Generators

We have

$$\frac{1}{2}(\tau_\xi \circ \sigma)\begin{pmatrix} 0 & P_n^T \\ P_n & 0 \end{pmatrix} = \sum_{m=0}^{n} \operatorname{tr}\left(\xi e^m \xi e^n f_-^{(n-m)}(e)\right),$$

$$(\tau_\xi \circ \sigma)\begin{pmatrix} 0 & P_n^T \\ P_{n+1} & 0 \end{pmatrix} = \sum_{m=0}^{n} \operatorname{tr}\Big(\xi e^m \xi \big(e^{n+1} f_-^{(n-m)}(e) + e^n f_-^{(n-m+1)}(e)\big)\Big).$$

Generators

The generators are $\mathrm{tr}(\xi e)$, $\mathrm{tr}(\xi e^2)$, ... and

$$\begin{aligned}
&\mathrm{tr}(\xi^2 e),\\
&\mathrm{tr}(2\xi^2 e^2 + \xi e \xi e),\\
&\mathrm{tr}(\xi^2 e^3 + \xi e \xi e^2),\\
&\mathrm{tr}(2\xi^2 e^4 + 2\xi e \xi e^3 + \xi e^2 \xi e^2 + \xi^2 e^2),\\
&\mathrm{tr}(\xi^2 e^5 + \xi e \xi e^4 + \xi e^2 \xi e^3 + \xi^2 e^3),\\
&\mathrm{tr}(2\xi^2 e^6 + 2\xi e \xi e^5 + 2\xi e^2 \xi e^4 + \xi e^3 \xi e^3 + 4\xi^2 e^4 + \xi e \xi e^3),\\
&\mathrm{tr}(\xi^2 e^7 + \xi e \xi e^6 + \xi e^2 \xi e^5 + \xi e^3 \xi e^4 + 3\xi^2 e^5 + \xi e \xi e^4), \ldots.
\end{aligned}$$

They are mutually commutative.

Deformation quantization

Suppose that M is a smooth manifold and let

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M), \qquad (x, y) \mapsto \{x, y\}$$

be a Poisson bracket. A *star product*

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\nu]],$$
$$(x, y) \mapsto x \star y = xy + \sum_{n=1}^{\infty} B_n(x, y)\nu^n$$

is called a deformation quantization if it satisfies the following.

Deformation quantization

1. $B_1, B_2, \ldots$ are bidifferential operators.
2. Associativity.
3. We have

$$B_1(x,y) - B_1(y,x) = \left[\frac{x \star y - y \star x}{\nu}\right]_{\nu=0} = \{x, y\}$$

for any smooth functions x and y.

Theorem (Kontsevich, 2003)

Any Poisson manifold has a deformation quantization.

Deformation quantization

The dual space g^* of a Lie algebra g is a Poisson manifold.

Remark

The image of the restriction of the star product on the product $Sg \times Sg$ is contained in the polynomial algebra $(Sg)[\nu]$.

It makes sense to put $\nu = 1$ and obtain the star product on the symmetric algebra Sg.

Theorem

The universal enveloping algebra Ug is isomorphic to the symmetric algebra Sg with the star product.

References

[1] Yasushi Ikeda. "Quasidifferential operator and quantum argument shift method". In: *Theoretical and Mathematical Physics* 212.1 (2022), pp. 918–924.

[2] Yasushi Ikeda. "Second-order quantum argument shifts in Ugl_d". In: *Theoretical and Mathematical Physics* 220.2 (2024), pp. 1294–1303.

[3] Yasushi Ikeda and Georgy Sharygin. "The argument shift method in universal enveloping algebra $U\mathfrak{gl}_d$". In: *Journal of Geometry and Physics* 195 (2024), p. 105030.

[4] Yasushi Ikeda, Alexander Molev, and Georgy Sharygin. "On the quantum argument shift method". In: *arXiv preprint arXiv:2309.15684* (2023).