北大 MCYR 20 March 4, 2024

Quantum analog of Mishchenko-Fomenko theorem for Ugl_d

$\mathfrak{g}$: Lie 環
$\|$
$\langle e_1, \ldots, e_d \rangle$ 基底

$\xi \in \mathfrak{g}^*$

$$\overline{\partial_\xi} = \xi(e_1)\frac{\partial}{\partial e_1} + \cdots + \xi(e_d)\frac{\partial}{\partial e_d}$$

$$\in \hom \boxed{S\mathfrak{g}} \quad \text{対称代数}$$

$$S\mathfrak{g} = \mathbb{C}[e_1, \ldots, e_d]$$

$(\mathfrak{g}^*)^*$
$\|$
$\underline{\mathfrak{g}} \subset \underline{S\mathfrak{g}} \subsetneq C^\infty(\mathfrak{g}^*)$

linear 多項式関数

Lie bracket + Leibniz

$$\|$$

$$\begin{tikzcd}
C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \arrow[rr, "\exists_1 \text{ Poisson bracket}"] & & C^\infty(\mathfrak{g}^*) \\
\mathfrak{g} \times \mathfrak{g} \arrow[u, phantom, "\cup" rotate=90] \arrow[rr, "\text{Lie bracket}"'] \arrow[r, phantom, "{[\ ,\ ]}"] & \circlearrowleft & \mathfrak{g} \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

Def 1 ↗ Lie-Poisson bracket

$\mathfrak{g}^*$ Poisson mfd → 変形量子化

Thm (M-F, 1978) $\overline{C} \subset S\mathfrak{g}$: Poisson center

$$\boxed{\overline{C}_\xi = \overline{C}[\partial_\xi x, \partial_\xi^2 x, \cdots : x \in \overline{C}]}$$

は Poisson 可換

$$\left(\begin{array}{l} S\mathfrak{g} \longrightarrow \text{変形量子化} \\ *\text{積},\ \underbrace{\nu = 1}_{\text{量子化パラメータ}} \end{array} \right) \simeq \underbrace{U\mathfrak{g}}_{S\mathfrak{g}\text{の変形量子化}}$$

$$\mathrm{gr}\, Ug = Sg$$ 3

$$\Downarrow$$

Def 2

C_ξ : ξにする量子 argument shift 代数

$$\underset{\mathrm{def}}{\Longleftrightarrow} \begin{cases} C_\xi \subset Ug : \text{可換部分代数} \\ \mathrm{gr}\, C_\xi = \overline{C}_\xi \end{cases}$$

$$\begin{tikzcd}
Sg & Ug \\
\overline{C}_\xi \arrow[r] \arrow[u, phantom, sloped, "\subset"] & C_\xi \arrow[u, phantom, sloped, "\subset"] \\
\overline{\partial}_\xi \arrow[r] \arrow[d, phantom, sloped, "\in"] & \partial_\xi \ ? \arrow[d, phantom, sloped, "\in"] \\
\mathrm{hom}\, Sg & \mathrm{hom}\, Ug
\end{tikzcd}$$

$$g = gl_d$$
$$= gl(d, \mathbb{C}) = \langle e_{ij} : i, j = 1, \dots, d \rangle$$

$$\begin{array}{ccccc} \mathfrak{gl}_d & \simeq & M(d,\mathbb{C}) & & \\ \downarrow & & \downarrow & & \\ e^i_j & \simeq & E^i_j & = j & \begin{array}{c} i \\ \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \end{array} \end{array}$$

(j,i)-行列単位

$$e = \begin{pmatrix} e^1_1 & \cdots & e^1_d \\ \vdots & & \vdots \\ e^d_1 & \cdots & e^d_d \end{pmatrix} \in M(d, \mathfrak{gl}_d)$$

$$\bar{\partial}^i_j = \frac{\partial}{\partial e^j_i} \in \hom S\mathfrak{gl}_d.$$

$$\begin{array}{cccc} \bar{\partial} : & S\mathfrak{gl}_d & \longrightarrow & M(d, S\mathfrak{gl}_d) \\ & \Downarrow & & \downarrow \\ & x & \longmapsto & \begin{pmatrix} \bar{\partial}^1_1 x & \cdots & \bar{\partial}^1_d x \\ \vdots & & \vdots \\ \bar{\partial}^d_1 x & \cdots & \bar{\partial}^d_d x \end{pmatrix} \end{array}$$

1. $\bar{\partial}^i_j \mathbb{C} = \{0\}$
2. $\bar{\partial}^i_j e^k_l = \delta^i_l \delta^k_j$.
3. $\bar{\partial}^i_j (xy) = (\bar{\partial}^i_j x) y + x (\bar{\partial}^i_j y)$

$Ug\ell_d$：非可換 ⇒ 1〜3 をみたす作用素ない．

3 $\longrightarrow$ $\partial^i_j(xy) = (\partial^i_j x)y + x(\partial^i_j y)$ $\boxed{+\sum_{k=1}^{d}(\partial^i_k x)(\partial^k_j y)}$

well-defined (Gurevich-Pyatov-Saponov)

$\xi \in g\ell_d^* \simeq M(d,\mathbb{C})$ $(\xi^i_j = \xi(e^j_i))$

$\bar{\partial}_\xi = \mathrm{tr}(\xi\bar{\partial}) \in \mathrm{hom}\, Sg\ell_d$

$\partial_\xi = \mathrm{tr}(\xi\partial) \in \mathrm{hom}\, Ug\ell_d$.

Thm (I-Sharygin) $C \subset Ug\ell_d$：center

$C_\xi = \mathbb{C}[\partial_\xi x, \partial_\xi^2 x, \dots \ x \in C]$

は ξ についての量子 argument shift alg.

(可換，$\mathrm{gr}\, C_\xi = \bar{C}_\xi$)

$\mathfrak{gl}_d^* \simeq M(d,\mathbb{C})$ で固有値全部異なる
$\longrightarrow$ 稠密

WMA $\xi = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_d \end{pmatrix}$ $\quad z_i \neq z_j \quad (i \neq j)$

Vinberg–Rybnikov

$\longrightarrow$ A_ξ : 可換, $\mathrm{gr}\, A_\xi = \overline{C_\xi}$ centraliser

$$\Big(\Big\{ e_i^i,\ q_i(\xi) = \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j} \Big\}_{i=1}^{d} \Big)$$

ISTS $C_\xi \subset A_\xi$ i.e.

$\forall i,\ \forall n,\ \forall x \in \mathbb{C}$, $\quad [e_i^i, \partial_\xi^n x] = [q_i(\xi), \partial_\xi^n x] = 0$

$$[\partial_\xi, \mathrm{ad}\, e_i^i] = 0$$
$$\| $$
$$[e_i^i, \cdot\ ]$$

direct comp.

Lem

$\{ x \in U\mathfrak{gl}_d : \mathrm{tr}(\xi[\partial q_c(\xi), \partial x]) = 0 \}$

は C を含む ∂_ξ の不変部分空間

$\partial x,\ x \in C$ の形 → テクニカルレポート

Thm の証明

$$n > 0,\ [q_c(\xi), \partial_\xi^{n-1} x] = 0$$

$$\begin{aligned}
\Rightarrow\ 0 &= \partial_\xi [q_c(\xi), \partial_\xi^{n-1} x] \\
&= \cancel{[\underbrace{\partial_\xi q_c(\xi)}_{\in C}, \partial_\xi^{n-1} x]} + [q_c(\xi), \partial_\xi^{n} x] \\
&\quad + \cancel{\mathrm{tr}(\xi[\partial q_c(\xi), \partial \partial_\xi^{n-1} x])} \quad \text{Lem} \\
&= [q_c(\xi), \partial_\xi^{n} x]
\end{aligned}$$

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