Constructing quantum argument-shift algebras via quantized shift operators

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Statement

Suppose that ξ is a $d \times d$ complex matrix and let

$$\partial_\xi = \mathrm{tr}(\xi\partial) = \sum_{i,j=1}^{d} \xi_i^j \partial_j^i,$$

where $\partial_j^i \in \hom U\mathfrak{gl}_d$ are the quantum derivations introduced by Gurevich, Pyatov, and Saponov. The main theorem is the following:

Theorem (I. and Sharygin, 2024)

Suppose that x and y are central elements of $U\mathfrak{gl}_d$. Then

$$\left[\partial_\xi^m x, \partial_\xi^n y\right] = 0$$

for any nonnegative integers m and n.

Symmetric and Universal Enveloping Algebras

Suppose that $\mathfrak{g}$ is a Lie algebra.

- Both the symmetric algebra $S\mathfrak{g}$ and the universal enveloping algebra $U\mathfrak{g}$ are associative algebras.
- The symmetric algebra $S\mathfrak{g}$ (commutative) gives classical descriptions and the universal enveloping algebra $U\mathfrak{g}$ (non-commutative) gives quantum descriptions of the Lie algebra $\mathfrak{g}$.
- The symmetric algebra $S\mathfrak{g} = \operatorname{gr} U\mathfrak{g}$ is the associated graded Poisson algebra of the universal enveloping algebra $U\mathfrak{g}$.

Symmetric and Universal Enveloping Algebras

- The symmetric algebra $S\mathfrak{g}$ is a subalgebra of the smooth functions algebra $C^\infty\mathfrak{g}^*$.
- There exists a unique Poisson bracket (Lie–Poisson bracket) on the smooth functions algebra $C^\infty\mathfrak{g}^*$ extending the Lie bracket.

$$\begin{tikzcd} C^\infty\mathfrak{g}^* \times C^\infty\mathfrak{g}^* \arrow[r, "\text{Lie–Poisson bracket}"] & C^\infty\mathfrak{g}^* \\ \mathfrak{g} \times \mathfrak{g} \arrow[u] \arrow[r, "\text{Lie bracket}"] & \mathfrak{g} \arrow[u] \end{tikzcd}$$

- The dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is a Poisson manifold.

Deformation Quantization

Consider a deformation quantization of $C^\infty \mathfrak{g}^*$.

Remark

$$\begin{tikzcd} C^\infty \mathfrak{g}^* \times C^\infty \mathfrak{g}^* \arrow[r, "\star"] & (C^\infty \mathfrak{g}^*)[[\nu]] \\ S\mathfrak{g} \times S\mathfrak{g} \arrow[r, "\star"] \arrow[u] & (S\mathfrak{g})[\nu] \arrow[u] \end{tikzcd}$$

- It makes sense to put $\nu = 1$ and obtain the star product on the symmetric algebra $S\mathfrak{g}$.
- The universal enveloping algebra $U\mathfrak{g}$ is isomorphic to the symmetric algebra $S\mathfrak{g}$ with the star product at $\nu = 1$.

Prior Research and Motivation

Take a basis $(e_n)_{n=1}^{d}$ of $\mathfrak{g}$ and let

$$\overline{\partial}_\xi = \sum_{n=1}^{d} \xi(e_n) \frac{\partial}{\partial e_n} \in \operatorname{der} S\mathfrak{g}$$

be the directional derivative along $\forall \xi \in \mathfrak{g}^*$. Let $\overline{C}$ be the Poisson center of $S\mathfrak{g}$. The following theorem is referred to as the argument shift method.

Theorem (A. Mishchenko and A. Fomenko, 1978)

The subset $\left\{ \overline{\partial}_\xi^n x : (n, x) \in \mathbb{N} \times \overline{C} \right\}$ *is Poisson commutative.*

Prior Research and Motivation

- We obtaine a $\boxed{\text{Poisson commutative}}$ subalgebra $\overline{C}_\xi$ generated by these elements $\overline{\partial}_\xi^n x$.
- Recall $\boxed{\operatorname{gr} U\mathfrak{g} = S\mathfrak{g}}$.
- Vinberg asked if the argument shift algebra $\overline{C}_\xi$ can be quantised to a $\boxed{\text{commutative}}$ subalgebra C_ξ of the universal enveloping algebra $U\mathfrak{g}$ in a way that $\boxed{\operatorname{gr} C_\xi = \overline{C}_\xi}$.
- Such C_ξ is called a quantum argument shift algebra.

Prior Research and Motivation

- **Vinberg's problem** has been resolved in two ways:
  - Twisted Yangians: Nazarov–Olshanski.
  - Symmetrisation mapping: Tarasov.
- Also resolved using the Feigin–Frenkel center:
  - for regular elements ξ: Feigin et al. and Rybnikov.
  - for simple Lie algebras of types A and C: Futorny–Molev and Molev–Yakimova.

Motivation

The purpose of my talk is to quantize not only the algebra $\overline{C}_\xi$ but also the **operator** $\overline{\partial}_\xi$.

Quantum Derivation

Let $e = \begin{pmatrix} e^1_1 & \dots & e^1_d \\ \dots & \dots & \dots \\ e^d_1 & \dots & e^d_d \end{pmatrix}$ be a matrix satisfying the following.

- The set
$$\left\{ e^i_j : i, j = 1, \dots, d \right\}$$
is a basis of the general linear Lie algebra $\mathfrak{gl}(d, \mathbb{C})$.
- We have the commutation relations
$$\left[e^{i_1}_{j_1}, e^{i_2}_{j_2} \right] = \delta^{i_1}_{j_2} e^{i_2}_{j_1} - \delta^{i_2}_{j_1} e^{i_1}_{j_2}.$$

Quantum Derivation

We define

$$\overline{\partial} x = \begin{pmatrix} \overline{\partial}^1_1 x & \dots & \overline{\partial}^1_d x \\ \dots & \dots & \dots \\ \overline{\partial}^d_1 x & \dots & \overline{\partial}^d_d x \end{pmatrix}, \qquad \overline{\partial}^i_j = \frac{\partial}{\partial e^j_i}$$

for any element x of the symmetric algebra $S\mathfrak{gl}(d, \mathbb{C})$.

Remark

The derivation

$$S\mathfrak{gl}_d \to M(d, S\mathfrak{gl}_d), \qquad x \mapsto \overline{\partial} x$$

is a unique linear mapping satisfying the following.

1. $\overline{\partial}\nu = 0$ for any scalar ν.
2. $\overline{\partial}\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (Leibniz rule)

   $$\overline{\partial}(xy) = (\overline{\partial}x)y + x(\overline{\partial}y)$$

   for any elements x and y of the symmetric algebra $S\mathfrak{gl}_d$.

Quantum Derivation

There is no such mapping on $U\mathfrak{gl}_d$ because it is <u>non-commutative</u>.

Quantum Derivation

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quantum derivation

$$\boxed{U\mathfrak{gl}_d} \to M(d, U\mathfrak{gl}_d), \qquad x \mapsto \partial x$$

is a unique linear mapping satisfying the following.

1. $\partial \nu = 0$ for any scalar ν.
2. $\partial\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (quantum Leibniz rule)

$$\partial(xy) = (\partial x)y + x(\partial y) + \boxed{(\partial x)(\partial y)}$$

for any elements x and y of the universal enveloping algebra $U\mathfrak{gl}_d$.

Quantum Derivation

Let C be the center of the algebra $U\mathfrak{gl}_d$. Suppose that ξ is a numerical matrix and let $\partial_\xi = \mathrm{tr}(\xi\partial)$. The main theorem is the following:

Theorem (I. and Sharygin, 2024)

The subset

$$\left\{ \partial_\xi^n x : (n, x) \in \mathbb{N} \times C \right\} \tag{1}$$

is commutative.

Corollary

The subalgebra C_ξ generated by the subset (1) is the quantum argument shift algebra in the direction ξ.

Quantum Derivation

- We may assume that $\xi = \mathrm{diag}(z_1, \ldots, z_d)$ is diagonal and $(z_1, \ldots, z_d)$ is distinct considering the adjoint action of the general linear Lie group GL_d.
- Vinberg and Rybnikov showed that the quantum argument shift algebra in the direction ξ is the centralizer of the set

$$\left\{ e_i^i, \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j} \right\}_{i=1}^{d}. \tag{2}$$

- Since, by definition, the quantum argument shift algebra is <u>commutative</u>, the proof is carried out by showing that the quantum argument shift $\partial_\xi^n x$ commutes with the elements (2) by induction on the natural number n.

Formula

The center C of the algebra $U\mathfrak{gl}_d$ is the free commutative algebra on the elements

$$\operatorname{tr} e, \qquad \dots, \qquad \operatorname{tr} e^d.$$

They are called the Gelfand invariants. We would like to calculate the quantum argument shift $\partial_\xi^n x$ for a central element x. It is necessary and even sufficient to calculate the quantum derivation $\partial(e^n)^i_j$.

Formula

Remark

The linear operator

$$U\mathfrak{gl}_d \to M(d, U\mathfrak{gl}_d), \qquad x \mapsto \mathrm{diag}(x, \dots, x) + \partial x$$

is an algebraic homomorphism and will be denoted by ∂ from now on. We have the *quantum Leibniz rule*

$$\partial(xy) = (\partial x)(\partial y)$$

for any elements x and y of the universal enveloping algebra $U\mathfrak{gl}_d$.

Formula

I obtained the following formula for the quantum derivation.

We define $f_{\pm}^{(n)}(x) = \sum_{m=0}^{n} \frac{1 \pm (-1)^{n-m}}{2} \binom{n-1}{m} x^m$.

Theorem (I, 2022)

We have

$$\begin{aligned} \partial(e^n)_j^i &= \sum_{m=0}^{n} \left(f_{+}^{(n-m)}(e)(e^m)_j^i + f_{-}^{(n-m)}(e)_j(e^m)^i\right) \\ &= \sum_{m=0}^{n} \left((e^m)_j^i f_{+}^{(n-m)}(e) + (e^m)_j f_{-}^{(n-m)}(e)^i\right). \end{aligned}$$

The formula is used for the base case.

Formula

We assume the following form

$$\partial(e^n)^i_j = \sum_{m=0}^{n} \left(g^{(n)}_m(e)(e^m)^i_j + h^{(n)}_m(e)_j(e^m)^i \right),$$

where $g^{(n)}_m$ and $h^{(n)}_m$ are polynomials. By the quantum Leibniz rule and the commutation relations

$$\left[(e^m)^i, e^k_j\right] = (e^m)^k \delta^i_j - \delta^k (e^m)^i_j,$$

We obtained the initial condition

$$g^{(0)}_0(x) = 1, \qquad h^{(0)}_0(x) = 0$$

Formula

and the recursive relation

$$
\begin{aligned}
g_0^{(n+1)}(x) &= \sum_{m=0}^{n} h_m^{(n)}(x) x^m, \\
g_m^{(n+1)}(x) &= g_{m-1}^{(n)}(x), && 0 < m \leq n+1, \\
h_m^{(n+1)}(x) &= g_m^{(n)}(x) + h_m^{(n)}(x) x, && 0 \leq m < n+1, \\
h_{n+1}^{(n+1)}(x) &= 0.
\end{aligned}
$$

Its solution is

$$
g_m^{(n)}(x) = f_+^{(n-m)}(x), \qquad h_m^{(n)}(x) = f_-^{(n-m)}(x).
$$

Formula

The inductive step reduces to proving

$$\left[\operatorname{ad} e_i^i, \partial_\xi\right] = \left[\left[\operatorname{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right], \partial_\xi\right] = 0.$$

It can be done by computation.

Generators

Our theorem causes a filtration

$$C_\xi^{(0)} = C, \qquad C_\xi^{(n)} = C_\xi^{(n-1)}\left[\partial_\xi^n C\right]$$

of the quantum argument shift algebra C_ξ. Using the formula we obtain

$$C_\xi^{(1)} = C_\xi^{(0)}\left[\operatorname{tr}\left(\xi e^n\right) : n = 1, 2, \ldots\right],$$

$$C_\xi^{(2)} = C_\xi^{(1)}\left[\tau_\xi \begin{pmatrix} 0 & P_n^\top \\ P_m & 0 \end{pmatrix} : m, n = 0, 1, 2, \ldots\right].$$

P_n: some matrix composed of binomial coefficients.

Generators

But as for the second line these generators are redundant:

$$C_\xi^{(2)} = C_\xi^{(1)}\left[\tau_\xi \begin{pmatrix} 0 & P_n^\top \\ P_m & 0 \end{pmatrix} : \boxed{|m-n| \leq 1}\right].$$

Lemma (I, 2025)

We have

$$\sigma \begin{pmatrix} 0 & P_m^\top \\ P_{m+2n} & 0 \end{pmatrix} = \sum_{k=0}^{n} \left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right) P_{m+k}^{(m+k)},$$

$$\sigma \begin{pmatrix} 0 & P_m^\top \\ P_{m+2n+1} & 0 \end{pmatrix} = \sum_{k=0}^{n} \binom{2n-k}{k} \left(P_{m+k+1}^{(m+k)} + P_{m+k}^{(m+k+1)} \right).$$

for any nonnegative integers m and n.

Generators

Lemma reduces to the following relations.

1. For $\varepsilon = 0, 1$,

$$\binom{x+y+n}{2n+\varepsilon} + \binom{x-y+n}{2n+\varepsilon} = \sum_{m=0}^{n} \binom{x+m}{2m+\varepsilon} \left(\binom{y+n-m}{2(n-m)} + \binom{y-1+n-m}{2(n-m)} \right).$$

2. $$\sum_{m=0}^{n} \binom{x-m}{m} \binom{y+m}{n-m} = \sum_{m=0}^{n} \binom{x+y-m}{m} \binom{m}{n-m}.$$

3. $$\binom{x}{n} = \sum_{m=0}^{n} \binom{x-m}{m} \binom{m}{n-m} + \sum_{m=0}^{n-1} \binom{x-1-m}{m} \binom{m}{n-1-m}.$$

Generators

They are shown by induction.
The generators are $\mathrm{tr}(\xi e)$, $\mathrm{tr}(\xi e^2)$, ... and

$$\begin{aligned}
&\mathrm{tr}(\xi^2 e),\\
&\mathrm{tr}(2\xi^2 e^2 + \xi e \xi e),\\
&\mathrm{tr}(\xi^2 e^3 + \xi e \xi e^2),\\
&\mathrm{tr}(2\xi^2 e^4 + 2\xi e \xi e^3 + \xi e^2 \xi e^2 + \xi^2 e^2),\\
&\mathrm{tr}(\xi^2 e^5 + \xi e \xi e^4 + \xi e^2 \xi e^3 + \xi^2 e^3),\\
&\mathrm{tr}(2\xi^2 e^6 + 2\xi e \xi e^5 + 2\xi e^2 \xi e^4 + \xi e^3 \xi e^3 + 4\xi^2 e^4 + \xi e \xi e^3),\\
&\mathrm{tr}(\xi^2 e^7 + \xi e \xi e^6 + \xi e^2 \xi e^5 + \xi e^3 \xi e^4 + 3\xi^2 e^5 + \xi e \xi e^4), \ldots .
\end{aligned}$$

They are mutually commutative.

Conclusions

- We consider a representation of another Lie algebra and obtain a quantum derivation/quantum argument shift operator. However, this naive operator does not satisfy the quantized argument shift method. This means that I still do not know the appropriate definition of the quantum derivation/quantum argument shift operator in the general case.
- It may be more promising to generalize this result to the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$. I am currently working on this.