Commutativity of Second-Order Quasiderivations in General Linear Lie Algebras

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Idea

$\mathfrak{g}$: Lie alg.

$S(\mathfrak{g}) \circlearrowleft \partial_\xi$ constant vector field

symmetric alg of $\mathfrak{g}$ $\Rightarrow$ Poisson alg.

M-F theorem

$$\{\partial_\xi^m(f), \partial_\xi^n(g)\} = 0$$

$$\forall f, \forall g \in Z_\pi(S(\mathfrak{g}))$$

quantisation

Idea

$$\mathfrak{g} = \mathfrak{gl}(d, \mathbb{C})$$

$$U(\mathfrak{g}) \circlearrowleft \partial_\xi$$

quasiderivation

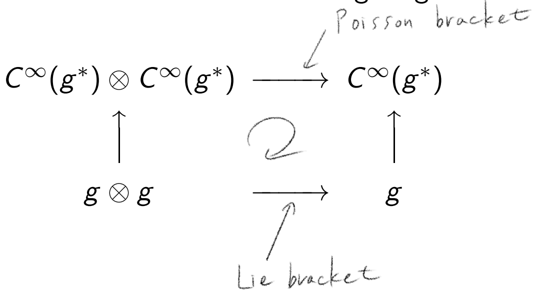
Conjecture

$$[\partial_\xi^m(f), \partial_\xi^n(g)] = 0$$

$$\forall f, \forall g \in Z(U(\mathfrak{g}))$$

Poisson Bracket on the Dual Space of a Lie Algebra

- We are going to investigate a quantum analogue of the theorem of A. Mishchenko and A. Fomenko.
- The dual space g^* of a finite dimensional real Lie algebra g is a Poisson manifold and the following diagram commutes.



Classical Theorem of A. Mishchenko and A. Fomenko

The classical theorem of A. Mishchenko and A. Fomenko is the following.[1]

Theorem (A. Mishchenko and A. Fomenko, 1978)

Suppose that ∂_ξ is a constant vector field on the dual space g^*. We have

$$\{\partial_\xi^m(x), \partial_\xi^n(y)\} = 0$$

for any m and n and for any Poisson central elements x and y of the symmetric algebra $S(g)$.

[1] Mishchenko and Fomenko, "Euler equations on finite-dimensional Lie groups".

Quantum Analogue of A. Mishchenko and A. Fomenko

We are going to investigate a quantum analogue of this theorem.

- We consider $g = gl(d, \mathbb{C})$.
- The symmetric algebra $S(g)$ should be replaced by the universal enveloping algebra $U(g)$.
- There is a candidate for a "derivation" of the universal enveloping algebra $U(g)$.

Quasiderivation of $Ugl(d, \mathbb{C})$

- Gurevich, Pyatov, and Saponov defined the quasiderivation of the universal enveloping algebra.[2]
- Let

$$e = \begin{pmatrix} e_1^1 & \cdots & e_d^1 \\ \vdots & \ddots & \vdots \\ e_1^d & \cdots & e_d^d \end{pmatrix} \in M\big(d, Ugl(d, \mathbb{C})\big),$$

where e_j^i form a linear basis of $gl(d, \mathbb{C})$ and satisfy the commutation relations $\left[e_j^i, e_l^k\right] = \delta_l^i e_j^k - \delta_j^k e_l^i$.

[2] Gurevich, Pyatov, and Saponov, "Braided Weyl algebras and differential calculus on $U(u(2))$".

Quasiderivation of $Ugl(d, \mathbb{C})$

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quasiderivation

$$Ugl(d, \mathbb{C}) \to M\big(d, Ugl(d, \mathbb{C})\big), \qquad x \mapsto \partial x$$

is a unique linear mapping satisfying the following.

1. We have $\partial \nu = 0$ for any scalar ν.
2. We have $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. We have the twisted Leibniz rule

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y)$$

for any elements x and y of the universal enveloping algebra.

Conjecture (Quantum Analogue)

Conjecture

Suppose that ξ is a numerical matrix and let $\partial_\xi = \mathrm{tr}(\xi\partial)$. We have

$$\left[\partial_\xi^m(x), \partial_\xi^n(y)\right] = 0$$

for any m and n and for any central elements x and y of the universal enveloping algebra $Ugl(d, \mathbb{C})$.

Poisson bracket on $S(g) \rightsquigarrow$ commutator on $U(g)$

Fundamental Formula and Corollary

We obtained a fundamental theorem for quasiderivations of central elements.[3]

Theorem (I, 2022)

We have the formula

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left((e^m)_j f_+^{(n-m-1)}(e)^i + (e^m)^i_j f_-^{(n-m-1)}(e) \right),$$

where we define the polynomials

$$f_\pm^{(n)}(x) = \frac{(x+1)^n \pm (x-1)^n}{2} = \sum_{m=0}^{n+1} \frac{1 \pm (-1)^{n-m}}{2} \binom{n}{m} x^m.$$

[3] Ikeda, "Quasidifferential operator and quantum argument shift method".

Fundamental Formula and Corollary

The center of the universal enveloping algebra

$$Z\bigl(Ugl(d,\mathbb{C})\bigr) \simeq \mathbb{C}\bigl[(\operatorname{tr} e^n)_{n=1}^{d}\bigr]$$

is a free commutative algebra on the set $\{\operatorname{tr} e^n\}_{n=1}^{d}$. We have

$$\partial_\xi(x) \in \operatorname{span}_Z\bigl\{\operatorname{tr}(\xi e^n)\bigr\}_{n=0}^{\infty}$$

for any central element x.

Corollary

The conjecture holds for $m = n = 1$. We have

$$\bigl[\partial_\xi(x), \partial_\xi(y)\bigr] = 0$$

for any central elements x and y.

Generators of Second-Order Quasiderivations

We consider the condition

$$\left[\partial_\xi(x), \partial_\xi^2(\operatorname{tr} e^n)\right] = 0 \tag{1}$$

for any central element x and for any n. We have

$$\partial_\xi^2(\operatorname{tr} e^n) = \sum_{m+\ell=0}^{n} \operatorname{tr}\Big(\xi\partial \operatorname{tr}\big(\xi f_-^{(\ell)}(e)\big) f_-^{(n-m-\ell-1)}(e)\Big) t_{m-1} + \operatorname{span}_Z\left\{\operatorname{tr}(\xi e^i)\operatorname{tr}(\xi e^j)\right\}_{i,j=0}^{\infty},$$

where we define $t_m = \begin{cases} -1, & m = -1 \\ \operatorname{tr} e^m, & m \geq 0 \end{cases}$. We have

$$(1) \Leftrightarrow \left[\operatorname{tr}(\xi e^k), \sum_{m=0}^{n+1} \operatorname{tr}\Big(\xi\partial \operatorname{tr}\big(\xi f_-^{(m-1)}(e)\big) f_-^{(n-m)}(e)\Big)\right] = 0, \quad \forall k, \forall n. \tag{2}$$

Generators of Second-Order Quasiderivations

The summation (2) is spanned by the set

$$\left\{ \operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^m)e^n\right) + \operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^n)e^m\right) \right\}_{m,n=0}^{\infty}.$$

Theorem

We have

$$\begin{aligned}
&\operatorname{span}_{\mathbb{Q}}\left\{ \operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^m)e^n\right) + \operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^n)e^m\right) \right\}_{m,n=0}^{\infty} \\
&\quad = \operatorname{span}_{\mathbb{Q}}\left\{ \partial_\xi \operatorname{tr}(\xi e^m) \right\}_{m=0}^{\infty} \\
&\quad = \operatorname{span}_{\mathbb{Q}}\Big\{ \operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^m)e^m\right), \\
&\qquad\qquad \operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^{m+1})e^m\right) + \operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^m)e^{m+1}\right) \Big\}_{m=0}^{\infty}
\end{aligned}$$

up to the subspace generated by the set $\left\{ \operatorname{tr}(\xi e^i) \operatorname{tr}(\xi e^j) \right\}_{i,j=0}^{\infty}$.

Generators of Second-Order Quasiderivations

Corollary

The following are equivalent.

1. *We have*
$$\left[\partial_\xi(x), \partial_\xi^2(\operatorname{tr} e^n)\right] = 0$$
for any central element x and for any n.
2. *The elements $\operatorname{tr}(\xi e^k)$ commute with the elements*
$$\operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^m)e^m\right),$$
$$\operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^{m+1})e^m\right) + \operatorname{tr}\left(\xi\partial \operatorname{tr}(\xi e^m)e^{m+1}\right).$$

Generators of Second-Order Quasiderivations

Theorem

We have

$$\begin{aligned}
&\operatorname{span}_{\mathbb{Q}}\left\{\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n+2m})e^{n}\right)+\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n})e^{n+2m}\right)\right\}_{m,n=0}^{\infty}\\
&\qquad=\operatorname{span}_{\mathbb{Q}}\left\{\partial_{\xi}\operatorname{tr}(\xi e^{2m})\right\}_{m=0}^{\infty}\\
&\qquad=\operatorname{span}_{\mathbb{Q}}\left\{\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{m})e^{m}\right)\right\}_{m=0}^{\infty},\\
&\operatorname{span}_{\mathbb{Q}}\left\{\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n+2m+1})e^{n}\right)+\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n})e^{n+2m+1}\right)\right\}_{m,n=0}^{\infty}\\
&\qquad=\operatorname{span}_{\mathbb{Q}}\left\{\partial_{\xi}\operatorname{tr}(\xi e^{2m+1})\right\}_{m=0}^{\infty}\\
&\qquad=\operatorname{span}_{\mathbb{Q}}\left\{\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{m+1})e^{m}\right)+\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{m})e^{m+1}\right)\right\}_{m=0}^{\infty}
\end{aligned}$$

up to the subspace generated by the set $\left\{\operatorname{tr}(\xi e^{i})\operatorname{tr}(\xi e^{j})\right\}_{i,j=0}^{\infty}$.

Key Matrix and Symmetry

Definition

1. We define P_n as the n by n submatrix of the following matrix.

$$\begin{pmatrix} \vdots \\ f_+^{(4)}(x) \\ f_+^{(3)}(x) \\ f_+^{(2)}(x) \\ f_+^{(1)}(x) \\ f_+^{(0)}(x) \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & 6 & 0 & 1 & \cdots \\ 0 & 3 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \\ \vdots \end{pmatrix}$$

P_5

2. We define $P_n^{(m)}$ as the matrix P_n shifted to the right by m positions.

Key Matrix and Symmetry

Suppose that A is a numerical square matrix.

Definition

We define

$$\begin{aligned}\varphi_\xi(A) &= \operatorname{tr}\left(\xi \quad \xi e \quad \cdots \quad \xi e^{n-1}\right) A \begin{pmatrix} \xi \\ \xi e \\ \vdots \\ \xi e^{n-1} \end{pmatrix} \\ &= \sum_{i,j=1}^{n} A^i_j \operatorname{tr}(\xi e^{i-1} \xi e^{j-1}).\end{aligned}$$

We have $\operatorname{tr}\left(\xi \partial \operatorname{tr}(\xi e^m) e^n\right) = \varphi_\xi\left(P^{(n)}_m\right)$.

Key Matrix and Symmetry

Definition

We define

$$\tau(A) = \begin{pmatrix} A_1^1 & A_2^1 + A_1^2 & \cdots & A_n^1 + A_1^n \\ 0 & A_2^2 & \cdots & A_n^2 + A_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^n \end{pmatrix}.$$

We have $\varphi_\xi(A) = \varphi_\xi\big(\tau(A)\big)$ since we have

$$\mathrm{tr}(\xi e^i \xi e^j) = \mathrm{tr}(\xi e^j \xi e^i).$$

Main Theorem (Matrix Form)

Theorem

We have

$$\tau\begin{pmatrix} 0 & P_{n+2m} \\ P_n^T & 0 \end{pmatrix} = \sum_{\ell=0}^{m}\left(\binom{2m-\ell}{\ell} + \binom{2m-\ell-1}{\ell-1}\right) P_{n+\ell}^{(n+\ell)}$$

and

$$\tau\begin{pmatrix} 0 & P_{n+2m+1} \\ P_n^T & 0 \end{pmatrix} = \sum_{\ell=0}^{m}\binom{2m-\ell}{\ell}\left(P_{n+\ell+1}^{(n+\ell)} + P_{n+\ell}^{(n+\ell+1)}\right).$$

Proof of Main Theorem (Even Case)

We have

$$
\begin{aligned}
&\operatorname{tr}\bigl(\xi\partial\operatorname{tr}(\xi e^{n+2m})e^{n}\bigr)+\operatorname{tr}\bigl(\xi\partial\operatorname{tr}(\xi e^{n})e^{n+2m}\bigr)\\
&\quad=\varphi_\xi\left(P_{n+2m}^{(n)}+P_n^{(n+2m)}\right)\\
&\quad=\varphi_\xi\left(\tau\begin{pmatrix}0 & P_{n+2m}\\ P_n^T & 0\end{pmatrix}\right)\\
&\quad=\sum_{\ell=0}^{m}\left(\binom{2m-\ell}{\ell}+\binom{2m-\ell-1}{\ell-1}\right)\varphi_\xi\bigl(P_{n+\ell}^{(n+\ell)}\bigr)\\
&\quad=\sum_{\ell=0}^{m}\left(\binom{2m-\ell}{\ell}+\binom{2m-\ell-1}{\ell-1}\right)\operatorname{tr}\bigl(\xi\partial\operatorname{tr}(\xi e^{n+\ell})e^{n+\ell}\bigr).
\end{aligned}
$$

Proof of Main Theorem (Odd Case)

Similarly,

$$\begin{aligned}
&\operatorname{tr}\bigl(\xi\partial\operatorname{tr}(\xi e^{n+2m+1})e^{n}\bigr)+\operatorname{tr}\bigl(\xi\partial\operatorname{tr}(\xi e^{n})e^{n+2m+1}\bigr)\\
&\qquad=\varphi_\xi\left(P^{(n)}_{n+2m+1}+P^{(n+2m+1)}_{n}\right)\\
&\qquad=\varphi_\xi\left(\tau\begin{pmatrix}0 & P_{n+2m+1}\\ P_n^T & 0\end{pmatrix}\right)\\
&\qquad=\sum_{\ell=0}^{m}\binom{2m-\ell}{\ell}\varphi_\xi\left(P^{(n+\ell)}_{n+\ell+1}+P^{(n+\ell+1)}_{n+\ell}\right)\\
&\qquad=\sum_{\ell=0}^{m}\binom{2m-\ell}{\ell}\\
&\qquad\qquad\Bigl(\operatorname{tr}\bigl(\xi\partial\operatorname{tr}(\xi e^{n+\ell+1})e^{n+\ell}\bigr)+\operatorname{tr}\bigl(\xi\partial\operatorname{tr}(\xi e^{n+\ell})e^{n+\ell+1}\bigr)\Bigr).
\end{aligned}$$

Equivalent Condition (Even Case)

The first part of the theorem is equivalent to the following.

1. We have

$$\binom{2n_1+n_2+2n_3+1}{2n_3}+\binom{n_2+2n_3}{2n_3}$$
$$=\sum_{n_4=0}^{n_3}\left(\binom{n_1+n_2+n_3+n_4+1}{2n_4}+\binom{n_1+n_2+n_3+n_4}{2n_4}\right)$$
$$\times\binom{n_1+n_3-n_4}{2(n_3-n_4)}$$

for any nonnegative integers $(n_k)_{k=1}^{3}$.

Equivalent Condition (Even Case)

2. We have

$$f_+^{(n+2m)}(x) + x^{2m} f_+^{(n)}(x) = \sum_{\ell=0}^{m} \left(\binom{2m-\ell}{\ell} + \binom{2m-\ell-1}{\ell-1} \right) x^\ell f_+^{(n+\ell)}(x)$$

for any nonnegative integer n.

$$f_+^{(n)}(x) = \frac{(x+1)^n + (x-1)^n}{2}.$$

These conditions have been verified using Mathematica.

Equivalent Condition (Odd Case)

The second part of the theorem is equivalent to the following.

1. We have

$$\binom{2n_1+n_2+2n_3+2}{2n_3}+\binom{n_2+2n_3}{2n_3} = \sum_{n_4=0}^{n_3}\binom{n_1+n_2+n_3+n_4+1}{2n_4} \times\left(\binom{n_1+n_3-n_4+1}{2(n_3-n_4)}+\binom{n_1+n_3-n_4}{2(n_3-n_4)}\right)$$

for any nonnegative integers $(n_k)_{k=1}^{3}$.

Equivalent Condition (Odd Case)

2. We have

$$f_{+}^{(n+2m+1)}(x) + x^{2m+1} f_{+}^{(n)}(x) = \sum_{\ell=0}^{m} \binom{2m-\ell}{\ell} \left(x^{\ell} f_{+}^{(n+\ell+1)}(x) + x^{\ell+1} f_{+}^{(n+\ell)}(x) \right)$$

for any nonnegative integer n.

$$f_{+}^{(n)}(x) = \frac{(x+1)^n + (x-1)^n}{2}.$$

These conditions have been verified using Mathematica.

Conclusion

- In a quantum analogue of the theorem of A. Mishchenko and A. Fomenko, the derivation of the symmetric algebra $Sgl(d, \mathbb{C})$ is replaced by the quasiderivation of the universal enveloping algebra $Ugl(d, \mathbb{C})$.
- We derived a concrete formula and proved the quantum analogue for order 1. Higher-order quasiderivations can be computed using this formula as well.
- Complex combinatorial formulas play a critical role in the calculations for the order 2 case. We verified these formulas using Mathematica. Uncovering the representation theory underlying these formulas should be the key to a general proof of the quantum analogue.

References

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