Lie Theory

Yasushi Ikeda

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Chapter 1

Introduction and Examples

1.1 Lie Algebras over Fields

Suppose that V is a vector space over a field $\mathbb{F}$.

Definition 1.1.1
We write $\hom V$ for the endomorphism algebra $\hom(V, V)$.

Proposition 1.1.1
The vector space $\hom V$ is a Lie algebra over $\mathbb{F}$.

Proof. The vector space $\hom V$ is an algebra over $\mathbb{F}$ and by Remark of 342 (cf. Lie Algebras). $\square$

Suppose that L is a Lie algebra over $\mathbb{F}$.

Definition 1.1.2
A representation of L on V is a homomorphism of L into $\hom V$.

Definition 1.1.3
We define $(\operatorname{ad} x)(y) = [x, y]$ for $\forall(x, y)$ of L^2.

Proposition 1.1.2
The mapping $x \mapsto \operatorname{ad} x$ is a representation of L on L.

Proof. The mapping $\operatorname{ad} x$ is a linear mapping on L for $\forall x$.

$$\begin{aligned}
[\operatorname{ad} x, \operatorname{ad} y](z) &= (\operatorname{ad} x)(\operatorname{ad} y)(z) - (\operatorname{ad} y)(\operatorname{ad} x)(z) \\
&= \big[x, [y, z]\big] - \big[y, [x, z]\big] \\
&= -\big[z, [x, y]\big] \\
&= \big(\operatorname{ad}[x, y]\big)(z)
\end{aligned}$$

for $\forall(x, y, z)$. $\square$

Definition 1.1.4
The mapping $x \mapsto \operatorname{ad} x$ is called the adjoint representation.

Definition 1.1.5
An invariant subspace for $\operatorname{ad}(L)$ is called an ideal.

1.2 The Killing Form on a Lie Algebra

Suppose that L is a finite dimensional Lie algebra over a field $\mathbb{F}$.

Definition 1.2.1
We define

$$B(x, y) = \operatorname{tr}\big((\operatorname{ad} x)(\operatorname{ad} y)\big)$$

for $\forall (x, y)$ of L^2. The symmetric form B is called the Killing form.

Proposition 1.2.1

1. The mapping

$$x \mapsto \big[y \mapsto B(x, y)\big] \tag{1.1}$$

is a linear mapping of L into L^*.

2. The Killing form on L is nondegenerate if and only if the linear mapping (1.1) is an isomorphism of L onto L^*.

Proposition 1.2.2
We have

$$B\big((\operatorname{ad} z)(x), y\big) = -B\big(x, (\operatorname{ad} z)(y)\big)$$

for $\forall (x, y, z)$ of L^3.

Proof. We have

$$\begin{aligned}
B\big((\operatorname{ad} z)(x), y\big) &= \operatorname{tr}\big(\operatorname{ad}[z, x] \operatorname{ad} y\big) \\
&= \operatorname{tr}\big([\operatorname{ad} z, \operatorname{ad} x] \operatorname{ad} y\big) \\
&= -\operatorname{tr}(\operatorname{ad} x \operatorname{ad} z \operatorname{ad} y - \operatorname{ad} z \operatorname{ad} x \operatorname{ad} y) \\
&= -\operatorname{tr}\big(\operatorname{ad} x(\operatorname{ad} z \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} z)\big) \\
&= -\operatorname{tr}\Big(\operatorname{ad} x \operatorname{ad}\big((\operatorname{ad} z)(y)\big)\Big) \\
&= -B\big(x, (\operatorname{ad} z)(y)\big).
\end{aligned}$$

□

We identify

$$M(2, \mathbb{F}) \leftrightarrow \mathbb{F}^4, \qquad \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \leftrightarrow \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix}.$$

Then we have

$$\begin{aligned}
\operatorname{ad}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} &= \operatorname{ad}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} -cx_{12} + bx_{21} & -bx_{11} + (a-d)x_{12} + bx_{22} \\ cx_{11} + (d-a)x_{21} - cx_{22} & cx_{12} - bx_{21} \end{pmatrix} \\
&= \begin{pmatrix} -cx_{12} + bx_{21} \\ -bx_{11} + (a-d)x_{12} + bx_{22} \\ cx_{11} + (d-a)x_{21} - cx_{22} \\ cx_{12} - bx_{21} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix}
\end{aligned}$$

for $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $M(2, \mathbb{F})$.

PROPOSITION 1.2.3 1. Suppose that $\{e_{ij}\}_{i,j=1}^{m}$ denote the matrix units. Then we have

$$\operatorname{ad}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(e_{11}, e_{12}, e_{21}, e_{22}) = (e_{11}, e_{12}, e_{21}, e_{22})\begin{pmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{pmatrix}$$

for $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $M(2, \mathbb{F})$.

2. The Killing form on the Lie algebra $M(2, \mathbb{F})$ is given by

$$\begin{aligned}
&B\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) \\
&\quad = \operatorname{tr}\left(\begin{pmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{pmatrix}\begin{pmatrix} 0 & -r & q & 0 \\ -q & p-s & 0 & q \\ r & 0 & s-p & -r \\ 0 & r & -q & 0 \end{pmatrix}\right) \\
&\quad = 4(br + cq) + 2(a-d)(p-s)
\end{aligned}$$

for $\forall \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)$ of $M(2, \mathbb{F})^2$.

3. We have

$$B\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 8bc + 2(a-d)^2$$

for $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $M(2, \mathbb{F})$.

Proposition 1.2.4
The set of traceless matrices

$$\{\, x \in M(m, \mathbb{F}) : \operatorname{tr} x = 0 \,\}$$

is an ideal of the Lie algebra $M(m, \mathbb{F})$ for $\forall m$ and

$$\dim\{\, x \in M(m, \mathbb{F}) : \operatorname{tr} x = 0 \,\} = m^2 - 1$$

for $\forall m \geq 1$.

Proof. The set $\{e_{ij}\}_{i \neq j} \cup \{e_{ii} - e_{i+1,i+1}\}_{i=1}^{m-1}$ is a basis of $\{\, x \in M(m, \mathbb{F}) : \operatorname{tr} x = 0 \,\}$ for $\forall m \geq 1$. □

We identify

$$\{\, x \in M(2, \mathbb{F}) : \operatorname{tr} x = 0 \,\} \leftrightarrow \mathbb{F}^3,$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = x_{11}H + x_{12}X + x_{21}Y \leftrightarrow \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \end{pmatrix},$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a basis of $\{\, x \in M(2, \mathbb{F}) : \operatorname{tr} x = 0 \,\}$. Then we have

$$\begin{aligned}
\operatorname{ad}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \end{pmatrix} &= \operatorname{ad}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \\
&= \begin{pmatrix} -cx_{12} + bx_{21} & -2bx_{11} + 2ax_{12} \\ 2cx_{11} - 2ax_{21} & cx_{12} - bx_{21} \end{pmatrix} \\
&= \begin{pmatrix} -cx_{12} + bx_{21} \\ -2bx_{11} + 2ax_{12} \\ 2cx_{11} - 2ax_{21} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \end{pmatrix}.
\end{aligned}$$

for $\forall \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ of $\{\, x \in M(2, \mathbb{F}) : \operatorname{tr} x = 0 \,\}$.

PROPOSITION 1.2.5 1. We have

$$\operatorname{ad}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}(H, X, Y) = (H, X, Y)\begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}$$

for $\forall \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ of $\left\{ x \in M(2, \mathbb{F}) : \operatorname{tr} x = 0 \right\}$.

2. The Killing form on the Lie algebra $\left\{ x \in M(2, \mathbb{F}) : \operatorname{tr} x = 0 \right\}$ is given by

$$\begin{aligned} &B\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} p & q \\ r & -p \end{pmatrix}\right) \\ &\qquad = \operatorname{tr}\left(\begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}\begin{pmatrix} 0 & -r & q \\ -2q & 2p & 0 \\ 2r & 0 & -2p \end{pmatrix}\right) \\ &\qquad = 4(br + cq) + 8ap \end{aligned}$$

for $\forall \left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} p & q \\ r & -p \end{pmatrix}\right)$ of $\left\{ x \in M(2, \mathbb{F}) : \operatorname{tr} x = 0 \right\}^2$.

PROPOSITION 1.2.6
The set of upper triangular matrices

$$\left\{ x \in M(m, \mathbb{F}) : x_{ij} = 0 \text{ for } \forall i > \forall j \right\}$$

is a Lie subalgebra of $M(m, \mathbb{F})$ for $\forall m$.

We identify

$$\left\{ x \in M(2, \mathbb{F}) : x_{21} = 0 \right\} \leftrightarrow \mathbb{F}^3, \qquad \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \leftrightarrow \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \operatorname{ad}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} &= \operatorname{ad}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -bx_{11} + (a - c)x_{12} + bx_{22} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -bx_{11} + (a - c)x_{12} + bx_{22} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ -b & a - c & b \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix}. \end{aligned}$$

for $\forall \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ of $\left\{ x \in M(2, \mathbb{F}) : x_{21} = 0 \right\}$.

Proposition 1.2.7

1. We have

$$\operatorname{ad}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}(e_{11}, e_{12}, e_{22}) = (e_{11}, e_{12}, e_{22})\begin{pmatrix} 0 & 0 & 0 \\ -b & a-c & b \\ 0 & 0 & 0 \end{pmatrix}$$

   for $\forall \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ of $\{ x \in M(2, \mathbb{F}) : x_{21} = 0 \}$.

2. The Killing form on the Lie algebra $\{ x \in M(2, \mathbb{F}) : x_{21} = 0 \}$ is given by

$$\begin{aligned} B\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\right) &= \operatorname{tr}\left(\begin{pmatrix} 0 & 0 & 0 \\ -b & a-c & b \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -q & p-r & q \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= (a-c)(p-r) \end{aligned}$$

   for $\forall \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\right)$ of $\{ x \in M(2, \mathbb{F}) : x_{21} = 0 \}^2$.

3. The Killing form on the Lie algebra $\{ x \in M(2, \mathbb{F}) : x_{21} = 0 \}$ is degenerate.

Proof.

$$B\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

for $\forall \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ of $\{ x \in M(2, \mathbb{F}) : x_{21} = 0 \}$. □

Proposition 1.2.8

The set of strictly upper triangular matrices

$$\{ x \in M(m, \mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \}$$

is a Lie subalgebra of $M(m, \mathbb{F})$ for $\forall m$.

We identify

$$\{ x \in M(3, \mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \} \leftrightarrow \mathbb{F}^3, \qquad \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} x_{12} \\ x_{13} \\ x_{23} \end{pmatrix}.$$

Then we have

$$\begin{aligned}
\operatorname{ad}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}\begin{pmatrix}x_{12}\\x_{13}\\x_{23}\end{pmatrix} &= \operatorname{ad}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}\begin{pmatrix}0&x_{12}&x_{13}\\0&0&x_{23}\\0&0&0\end{pmatrix}\\
&= \begin{pmatrix}0&0&-cx_{12}+ax_{23}\\0&0&0\\0&0&0\end{pmatrix}\\
&= \begin{pmatrix}0\\-cx_{12}+ax_{23}\\0\end{pmatrix}\\
&= \begin{pmatrix}0&0&0\\-c&0&a\\0&0&0\end{pmatrix}\begin{pmatrix}x_{12}\\x_{13}\\x_{23}\end{pmatrix}
\end{aligned}$$

for $\forall \begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}$ of $\left\{\, x \in M(3,\mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \,\right\}$.

PROPOSITION 1.2.9

1. We have

$$\operatorname{ad}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}(e_{12},e_{13},e_{23}) = (e_{12},e_{13},e_{23})\begin{pmatrix}0&0&0\\-c&0&a\\0&0&0\end{pmatrix}$$

   for $\forall \begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}$ of $\left\{\, x \in M(3,\mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \,\right\}$.

2. The Killing form on the Lie algebra $\left\{\, x \in M(3,\mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \,\right\}$ is given by

$$B\left(\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix},\begin{pmatrix}0&p&q\\0&0&r\\0&0&0\end{pmatrix}\right) = \operatorname{tr}\left(\begin{pmatrix}0&0&0\\-c&0&a\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\-r&0&p\\0&0&0\end{pmatrix}\right) = 0$$

   for $\forall \left(\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix},\begin{pmatrix}0&p&q\\0&0&r\\0&0&0\end{pmatrix}\right)$ of $\left\{\, x \in M(3,\mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \,\right\}^2$.

3. The Killing form on the Lie algebra $\left\{\, x \in M(3,\mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \,\right\}$ is degenerate.

PROPOSITION 1.2.10

The set of alternating matrices

$$\left\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\right\}$$

is a Lie subalgebra of $M(m,\mathbb{F})$ for $\forall m$.

Proposition 1.2.11
We have

$$B(x,y) = 2m\operatorname{tr}(xy) - 2(\operatorname{tr}x)(\operatorname{tr}x)$$

for $\forall x$ of $M(m,\mathbb{F})$.

Proof. Suppose that A is an algebra over $\mathbb{F}$ and let (x,y) be an element of A^2.

$$\begin{aligned}(\operatorname{ad}x)(\operatorname{ad}y) &= \big(L(x)-R(x)\big)\big(L(y)-R(y)\big)\\ &= L(xy) - L(x)R(y) - R(x)L(y) + R(yx).\end{aligned}$$

Suppose that A is finite dimensional.

$$B(x,y) = \operatorname{tr}L(xy) - \operatorname{tr}\big(L(x)R(y)\big) - \operatorname{tr}\big(R(x)L(y)\big) + \operatorname{tr}R(yx).$$

Suppose that $A = M(m,\mathbb{F})$. Since

$$\begin{aligned}\operatorname{tr}L(x) &= \sum_{i,j=1}^{m}(xe_{ij})_{ij} = \sum_{i,j=1}^{m}x_{ii} = m\operatorname{tr}x\\ \operatorname{tr}\big(L(x)R(y)\big) &= \sum_{i,j=1}^{m}(xe_{ij}y)_{ij} = \sum_{i,j=1}^{m}x_{ii}y_{jj} = (\operatorname{tr}x)(\operatorname{tr}y)\\ \operatorname{tr}R(x) &= \sum_{i,j=1}^{m}(e_{ij}x)_{ij} = \sum_{i,j=1}^{m}x_{jj} = m\operatorname{tr}x,\end{aligned}$$

we have $B(x,y) = 2m\operatorname{tr}(xy) - 2(\operatorname{tr}x)(\operatorname{tr}y)$. □

Example 1.2.1
We have

$$\begin{aligned}B\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix},\begin{pmatrix}p & q\\ r & s\end{pmatrix}\right) &= 4\operatorname{tr}\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}\right) - 2\operatorname{tr}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\operatorname{tr}\begin{pmatrix}p & q\\ r & s\end{pmatrix}\\ &= 4(ap+br+cq+ds) - 2(a+d)(p+s)\\ &= 4(br+cq) + 2(a-d)(p-s)\end{aligned}$$

for $\forall\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix},\begin{pmatrix}p & q\\ r & s\end{pmatrix}\right)$ of $M(2,\mathbb{F})^2$.

Proposition 1.2.12
Suppose that L_0 is an ideal of L.

1. The vector space L/L_0 is a Lie algebra.

2. The Killing form on L_0 is the restriction of the Killing form on L.

Proof. There exists a basis $\{e_k\}_{k=1}^n$ of L such that $\{e_k\}_{k=1}^{n_0}$ is a basis of L_0. We define $\{f_k\}_{k=1}^n$ to be the dual basis of $\{e_k\}_{k=1}^n$.

$$\begin{aligned} B(x,y) &= \sum_{k=1}^{n} f_k\big((\operatorname{ad} x)(\operatorname{ad} y)(e_k)\big) \\ &= \sum_{k=1}^{n_0} f_k\big((\operatorname{ad} x)(\operatorname{ad} y)(e_k)\big) \end{aligned}$$

for $\forall(x,y)$ of L_0^2. □

EXAMPLE 1.2.2 1. The Killing form on the ideal $\{\, x \in M(m,\mathbb{F}) : \operatorname{tr} x = 0 \,\}$ of the Lie algebra $M(m,\mathbb{F})$ is given by

$$B(x,y) = 2m \operatorname{tr}(xy)$$

for $\forall(x,y)$ of $\{\, x \in M(m,\mathbb{F}) : \operatorname{tr} x = 0 \,\}^2$.

2. The Killing form on the ideal $\{\, x \in M(2,\mathbb{F}) : \operatorname{tr} x = 0 \,\}$ of the Lie algebra $M(2,\mathbb{F})$ is given by

$$\begin{aligned} B\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} p & q \\ r & -p \end{pmatrix}\right) &= 4 \operatorname{tr}\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} p & q \\ r & -p \end{pmatrix}\right) \\ &= 4(br + cq) + 8ap \end{aligned}$$

for $\forall\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} p & q \\ r & -p \end{pmatrix}\right)$ of $\{\, x \in M(2,\mathbb{F}) : \operatorname{tr} x = 0 \,\}^2$.

PROPOSITION 1.2.13
Suppose that J is an element of $M(m,\mathbb{F})$.

$$L = \{\, x \in M(m,\mathbb{F}) : x^T J + Jx = 0 \,\}$$

is a Lie subalgebra of $M(m,\mathbb{F})$.

Proof. Suppose that x and y are elements of L.

$$\begin{aligned} [x,y]^T J + J[x,y] &= \left(y^T x^T - x^T y^T\right) J + J[x,y] \\ &= -y^T Jx + x^T Jy + J[x,y] \\ &= Jyx - Jxy + J[x,y] \\ &= 0 \end{aligned}$$

and the element $[x,y]$ belongs to L. □

PROPOSITION 1.2.14

$$\dim \left\{ x \in M(2m,\mathbb{F}) : x^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x = 0 \right\} = 2m^2 + m.$$

Proof.

$$\begin{aligned} x^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x = 0 \\ \Leftrightarrow \begin{pmatrix} x_{11}^T & x_{21}^T \\ x_{12}^T & x_{22}^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = 0 \\ \Leftrightarrow \begin{pmatrix} -x_{21}^T + x_{21} & x_{11}^T + x_{22} \\ -x_{22}^T - x_{11} & x_{12}^T - x_{12} \end{pmatrix} = 0 \end{aligned}$$

for $\forall x$ of $M(2m, \mathbb{F})$.

The set

$$\begin{aligned} \left\{ e_{ij} - e_{m+j,m+i} \right\}_{i,j=1}^{m} \cup \left\{ e_{i,m+i} \right\}_{i=1}^{m} \cup \left\{ e_{i,m+j} + e_{j,m+i} \right\}_{1 \le i < j \le m} \\ \cup \left\{ e_{m+i,i} \right\}_{i=1}^{m} \cup \left\{ e_{m+i,j} + e_{m+j,i} \right\}_{1 \le i < j \le m} \end{aligned}$$

is a basis of the Lie algebra. □

1.3 Examples of the Killing Forms

Suppose that $\mathbb{F}$ is a field such that $\operatorname{char}(\mathbb{F}) \neq 2$.

Proposition 1.3.1
The Killing form on the ideal $\left\{ x \in M(2, \mathbb{F}) : \operatorname{tr} x = 0 \right\}$ of the Lie algebra $M(2, \mathbb{F})$ is nondegenerate.

Proof. Suppose that x is an element of $\left\{ x \in M(2, \mathbb{F}) : \operatorname{tr} x = 0 \right\}$ such that $\left[y \mapsto B(x, y) \right] = 0$. Then we have

$$x = B(x, 8^{-1}H)H + B(x, 4^{-1}Y)X + B(x, 4^{-1}X)Y = 0. \quad \square$$

Proposition 1.3.2
We have

$$B(x, y) = \frac{B(x+y, x+y) - B(x, x) - B(y, y)}{2}$$

for $\forall (x, y)$ of L^2.

We identify

$$\left\{ x \in M(3, \mathbb{F}) : x^T = -x \right\} \leftrightarrow \mathbb{F}^3, \qquad \begin{pmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & x_{23} \\ -x_{13} & -x_{23} & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} x_{12} \\ x_{13} \\ x_{23} \end{pmatrix}.$$

Then we have

$$\begin{aligned}
\operatorname{ad}\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{13} \\ x_{23} \end{pmatrix} &= \operatorname{ad}\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}\begin{pmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & x_{23} \\ -x_{13} & -x_{23} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & cx_{13} - bx_{23} & -cx_{12} + ax_{23} \\ -cx_{13} + bx_{23} & 0 & bx_{12} - ax_{13} \\ cx_{12} - ax_{23} & -bx_{12} + ax_{13} & 0 \end{pmatrix} \\
&= \begin{pmatrix} cx_{13} - bx_{23} \\ -cx_{12} + ax_{23} \\ bx_{12} - ax_{13} \end{pmatrix} \\
&= \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{13} \\ x_{23} \end{pmatrix}
\end{aligned}$$

for $\forall \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ of $\left\{ x \in M(3, \mathbb{F}) : x^T = -x \right\}$.

Proposition 1.3.3 1. We have

$$\begin{aligned}
&\operatorname{ad}\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}(e_{12} - e_{21}, e_{13} - e_{31}, e_{23} - e_{32}) \\
&\qquad = (e_{12} - e_{21}, e_{13} - e_{31}, e_{23} - e_{32})\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}
\end{aligned}$$

for $\forall \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ of $\left\{ x \in M(3, \mathbb{F}) : x^T = -x \right\}$.

2. The Killing form on the Lie algebra $\left\{ x \in M(3, \mathbb{F}) : x^T = -x \right\}$ is given by

$$\begin{aligned}
&B\left(\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \begin{pmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{pmatrix}\right) \\
&\qquad = \operatorname{tr}\left(\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}\begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix}\right) = -2(ap + bq + cr)
\end{aligned}$$

for $\forall \left(\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \begin{pmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{pmatrix}\right)$ of $\left\{ x \in M(3, \mathbb{F}) : x^T = -x \right\}^2$.

3. We have

$$B\left(\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}\right) = -2\big(a^2+b^2+c^2\big)$$

for $\forall \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ of $\{\, x \in M(3,\mathbb{F}) : x^T = -x \,\}$.

Proposition 1.3.4 1. We have

$$\dim\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\} = \frac{m(m-1)}{2}.$$

2. The Killing form on the Lie algebra $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}$ is given by

$$B(x,y) = (m-2)\operatorname{tr}(xy)$$

for $\forall (x,y)$ of $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}^2$.

3. The Killing form on the Lie algebra $\{\, x \in M(3,\mathbb{F}) : x^T = -x \,\}$ is given by

$$\begin{aligned} &B\left(\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \begin{pmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{pmatrix}\right) \\ &\qquad = \operatorname{tr}\left(\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{pmatrix}\right) = -2(ap+bq+cr) \end{aligned}$$

for $\forall \left(\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \begin{pmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{pmatrix}\right)$ of $\{\, x \in M(3,\mathbb{F}) : x^T = -x \,\}^2$.

Proof. The set $\{e_{ij} - e_{ji}\}_{i<j}$ is a basis of $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}$. Suppose that x is an element of $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}$.

$$\begin{aligned} B(x,x) &= \sum_{i<j} \big((\operatorname{ad} x)^2(e_{ij} - e_{ji})\big)_{ij} \\ &= \sum_{i<j} \big(x^2(e_{ij}-e_{ji}) - 2x(e_{ij}-e_{ji})x + (e_{ij}-e_{ji})x^2\big)_{ij} \\ &= \sum_{i<j} \big((x^2)_{ii} + 2x_{ij}^2 + (x^2)_{jj}\big) \\ &= (m-1)\operatorname{tr} x^2 + 2\sum_{i<j} x_{ij}^2, \end{aligned}$$

where

$$2\sum_{i<j} x_{ij}^2 = \sum_{i<j} x_{ij}^2 + \sum_{j<i} x_{ji}^2 = \sum_{i<j} x_{ij}^2 + \sum_{j<i} x_{ij}^2$$
$$= \sum_{i\neq j} x_{ij}^2 = \sum_{i,j=1}^{m} x_{ij}^2 = -\sum_{i,j=1}^{m} x_{ij}x_{ji} = -\operatorname{tr} x^2.$$

$$B(x,x) = (m-2)\operatorname{tr} x^2.$$

$$B(x,y) = (m-2)\operatorname{tr}(xy)$$

for $\forall(x,y)$ of $\left\{ x \in M(m,\mathbb{F}) : x^T = -x \right\}^2$. □

EXAMPLE 1.3.1 1. The Killing form on the Lie algebra $\left\{ x \in M(2,\mathbb{F}) : x^T = -x \right\}$ is not the restriction of the Killing form on the Lie algebra $\left\{ x \in M(2,\mathbb{F}) : \operatorname{tr} x = 0 \right\}$.

2. The Lie algebra $\left\{ x \in M(2,\mathbb{F}) : x^T = -x \right\}$ is not an ideal of the Lie algebra $\left\{ x \in M(2,\mathbb{F}) : \operatorname{tr} x = 0 \right\}$.

Proof. We have

$$\begin{aligned} B_{M(2,\mathbb{F})}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) &= 4\operatorname{tr}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 \\ &= -4\operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= -8 \neq 0. \end{aligned}$$ □

PROPOSITION 1.3.5
The Killing form on the Lie algebra

$$L = \left\{ x \in M(2m,\mathbb{F}) : x^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x = 0 \right\}$$

is given by

$$B(x,y) = 2(m+1)\operatorname{tr}(xy)$$

for $\forall(x,y)$ of L^2.

Proof. Suppose that x is an element of L.

$$\begin{aligned}
F = &\sum_{i,j=1}^{m} \big(x^2(e_{ij} - e_{m+j,m+i})\big)_{ij} \\
&\quad + \sum_{i=1}^{m}(x^2 e_{i,m+i})_{i,m+i} + \sum_{1\le i<j\le m} \big(x^2(e_{i,m+j} + e_{j,m+i})\big)_{i,m+j} \\
&\quad + \sum_{i=1}^{m}(x^2 e_{m+i,i})_{m+i,i} + \sum_{1\le i<j\le m} \big(x^2(e_{m+i,j} + e_{m+j,i})\big)_{m+i,j} \\
&\quad = \sum_{i,j=1}^{m} (x^2 e_{ij})_{ij} + \sum_{i=1}^{m}\big((x^2 e_{i,m+i})_{i,m+i} + (x^2 e_{m+i,i})_{m+i,i}\big) \\
&\qquad + \sum_{1\le i<j\le m} \big((x^2 e_{i,m+j})_{i,m+j} + (x^2 e_{m+i,j})_{m+i,j}\big) \\
&\qquad = \sum_{i,j=1}^{m} (x^2)_{ii} + \sum_{1\le i\le j\le m} \big((x^2)_{ii} + (x^2)_{m+i,m+i}\big) \\
&\qquad\quad = m\,\mathrm{tr}(x_{11}^2 + x_{12}x_{21}) + \sum_{1\le i\le j\le m} \big((x^2)_{ii} + (x^2)_{m+i,m+i}\big).
\end{aligned}$$

$$\begin{aligned}
H = &\sum_{i,j=1}^{m} (e_{ij}x^2)_{ij} + \sum_{i=1}^{m}\big((e_{i,m+i}x^2)_{i,m+i} + (e_{m+i,i}x^2)_{m+i,i}\big) \\
&\qquad + \sum_{1\le i<j\le m} \big((e_{i,m+j}x^2)_{i,m+j} + (e_{m+i,j}x^2)_{m+i,j}\big) \\
&\qquad = \sum_{i,j=1}^{m} (x^2)_{jj} + \sum_{1\le i\le j\le m} \big((x^2)_{m+j,m+j} + (x^2)_{jj}\big) \\
&\qquad\quad = m\,\mathrm{tr}(x_{11}^2 + x_{12}x_{21}) + \sum_{1\le i\le j\le m} \big((x^2)_{m+j,m+j} + (x^2)_{jj}\big).
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(x^2) &= \mathrm{tr}(x_{11}^2 + x_{12}x_{21}) + \mathrm{tr}(x_{21}x_{12} + x_{22}^2) \\
&= 2\,\mathrm{tr}(x_{11}^2 + x_{12}x_{21}).
\end{aligned}$$

$$\begin{aligned}
F + H &= m\,\mathrm{tr}(x^2) + (m+1)\,\mathrm{tr}(x^2) \\
&= (2m+1)\,\mathrm{tr}(x^2).
\end{aligned}$$

$$\begin{aligned}
G = \sum_{i,j=1}^{m} (x_{ii}x_{jj} - x_{i,m+j}x_{m+i,j}) \\
+ \sum_{i=1}^{m} x_{ii}x_{m+i,m+i} + \sum_{1\le i<j\le m} (x_{ii}x_{m+j,m+j} + x_{ij}x_{m+i,m+j}) \\
+ \sum_{i=1}^{m} x_{m+i,m+i}x_{ii} + \sum_{1\le i<j\le m} (x_{m+i,m+i}x_{jj} + x_{m+i,m+j}x_{ij}) \\
= -\sum_{i,j=1}^{m} x_{ij}x_{ji} - \sum_{i,j=1}^{m} (x_{12})_{ij}(x_{21})_{ij} = -\operatorname{tr}(x_{11}^2 + x_{12}x_{21}).
\end{aligned}$$

$$B(x,x) = F + H - 2G = 2(m+1)\operatorname{tr}(x^2).$$

$$B(x,y) = 2(m+1)\operatorname{tr}(xy)$$

for $\forall(x,y)$ of L^2. □

PROPOSITION 1.3.6
The Killing form on $M(m,\mathbb{F})$ is degenerate for $\forall m \ge 1$.

Proof.

$$B(x,1) = 2m\operatorname{tr} x - 2(\operatorname{tr} x)(\operatorname{tr} 1) = 0$$

for $\forall x$ of $M(m,\mathbb{F})$. □

1.4 Lie Algebras over Fields of Characteristic Two

Suppose that $\mathbb{F}$ is a field of characteristic 2.

PROPOSITION 1.4.1 1. We have

$$\dim\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\} = \frac{m(m+1)}{2}.$$

2. The Killing form on the Lie algebra $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}$ is given by

$$B(x,y) = m\operatorname{tr}(xy) - (\operatorname{tr} x)(\operatorname{tr} y)$$

for $\forall(x,y)$ of $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}^2$.

Proof. The set $\{e_{ij}+e_{ji}\}_{i<j} \cup \{e_{ii}\}_{i=1}^m$ is a basis of $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}$. Suppose that (x,y) is an element of $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}^2$.

$$\begin{aligned}
B(x,y) &= \operatorname{tr}\big((\operatorname{ad} x)(\operatorname{ad} y)\big) \\
&= \sum_{i<j} \big((\operatorname{ad} x)(\operatorname{ad} y)(e_{ij}+e_{ji})\big)_{ij} + \sum_{i=1}^{m} \big((\operatorname{ad} x)(\operatorname{ad} y)(e_{ii})\big)_{ii}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{i<j}\big((\operatorname{ad} x)(\operatorname{ad} y)(e_{ij}+e_{ji})\big)_{ij} \\
&\quad = \sum_{i<j}\big(xy(e_{ij}+e_{ji}) - x(e_{ij}+e_{ji})y - y(e_{ij}+e_{ji})x + (e_{ij}+e_{ji})yx\big)_{ij} \\
&\qquad = \sum_{i<j}\big((xy)_{ii} - x_{ii}y_{jj} - y_{ii}x_{jj} + (yx)_{jj}\big) \\
&\qquad\quad = \sum_{i\neq j}\big((xy)_{ii} - x_{ii}y_{jj}\big) = m\operatorname{tr}(xy) - (\operatorname{tr} x)(\operatorname{tr} y).
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{m}\big((\operatorname{ad} x)(\operatorname{ad} y)(e_{ii})\big)_{ii} &= \sum_{i=1}^{m}(xye_{ii} - xe_{ii}y - ye_{ii}x + e_{ii}yx)_{ii} \\
&= \sum_{i=1}^{m}\big((xy)_{ii} + (yx)_{ii}\big) = 0.
\end{aligned}$$

$$B(x,y) = m\operatorname{tr}(xy) - (\operatorname{tr} x)(\operatorname{tr} y). \quad \square$$

1.5 Lie Algebras over Fields of Characteristic Zero

Suppose that $\mathbb{F}$ is a field of characteristic 0.

THEOREM 1.5.1
The Killing form on $\{\, x \in M(m,\mathbb{F}) : \operatorname{tr} x = 0 \,\}$ is nondegenerate.

Proof. We may assume that $m \geq 1$. Suppose that x is an element of $\{\, x \in M(m,\mathbb{F}) : \operatorname{tr} x = 0 \,\}$ such that $\big[y \mapsto B(x,y)\big] = 0$.

$$x_{ij} = \operatorname{tr}(xe_{ji}) = 0$$

for $\forall i \neq \forall j$ and we have

$$x_{11} = \cdots = x_{mm} = \frac{\operatorname{tr} x}{m} = 0$$

since

$$x_{ii} - x_{i+1,i+1} = \operatorname{tr}\big(x(e_{ii} - e_{i+1,i+1})\big) = 0$$

for $\forall i < m$. $\square$

THEOREM 1.5.2
The Killing form on $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}$ is nondegenerate for $\forall m \geq 3$.

Proof. Suppose that x is an element of $\{\, x \in M(m,\mathbb{F}) : x^T = -x \,\}$ such that $\big[y \mapsto B(x,y)\big] = 0$.

$$x_{ij} = \frac{x_{ij} - x_{ji}}{2} = -\frac{\operatorname{tr}\big(x(e_{ij} - e_{ji})\big)}{2} = 0$$

for $\forall i < \forall j$. $\square$

THEOREM 1.5.3

The Killing form on

$$L = \left\{ x \in M(2m, \mathbb{F}) : x^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x = 0 \right\}$$

is nondegenerate.

Proof. Suppose that x is an element of L such that $\big[y \mapsto B(x,y)\big] = 0$.

$$\begin{aligned} x_{ij} &= \frac{x_{ij} - x_{m+j,m+i}}{2} = \frac{\operatorname{tr}\big(x(e_{ji} - e_{m+i,m+j})\big)}{2} = 0 \\ x_{i,m+j} &= \frac{x_{j,m+i} + x_{i,m+j}}{2} = \frac{\operatorname{tr}\big(x(e_{m+i,j} + e_{m+j,i})\big)}{2} = 0 \\ x_{m+i,j} &= \frac{x_{m+j,i} + x_{m+i,j}}{2} = \frac{\operatorname{tr}\big(x(e_{i,m+j} + e_{j,m+i})\big)}{2} = 0 \end{aligned}$$

for $\forall (i,j)$ of $\big\{1, \ldots, m\big\}^2$. $\square$

Chapter 2

Fundamental Theorems

2.1 Engel's Theorem

Definition 2.1.1

A Lie algebra L is said to be nilpotent if eventually

$$L, [L, L], \big[L, [L, L]\big], \dots, \{0\}, \dots.$$

Proposition 2.1.1

A Lie algebra L is nilpotent if and only if

$$\big\{ (\operatorname{ad} x_1) \cdots (\operatorname{ad} x_n) : (x_k)_{k=1}^n \in L^n \big\} = \{0\}$$

for some n.

Proposition 2.1.2

Suppose that L is a nilpotent Lie algebra. Then the set $\operatorname{ad} L$ is a subset of

$$\big\{ x \in \hom L : x \text{ is nilpotent} \big\} = \big\{ x \in \hom L : x^n = 0 \text{ for some } n \big\}.$$

Proposition 2.1.3

Suppose that V_0 is a subspace of a vector space V.

1. $\big\{e_k + V_0\big\}_{k=n_0+1}^n$ is a basis of V/V_0 if $\big\{e_k\big\}_{k=1}^n$ is a basis of V such that $\big\{e_k\big\}_{k=1}^{n_0}$ is a basis of V_0.
2. $\big\{e_k\big\}_{k=1}^n$ is a basis of V if $\big\{e_k\big\}_{k=1}^{n_0}$ is a basis of V_0 and $\big\{e_k + V_0\big\}_{k=n_0+1}^n$ is a basis of V/V_0.

Proof. Suppose that V is a vector space over a field $\mathbb{F}$.

1. Suppose that $\big\{e_k\big\}_{k=1}^n$ is a basis of V such that $\big\{e_k\big\}_{k=1}^{n_0}$ is a basis of V_0.

   Suppose that $\big\{\nu_k\big\}_{k=n_0+1}^n$ is a subset of $\mathbb{F}$ such that

$$\sum_{k=n_0+1}^{n} \nu_k (e_k + V_0) = 0.$$

The vector
$$\sum_{k=n_0+1}^{n} \nu_k e_k$$
belongs to V_0 and we have $\{\nu_k\}_{k=n_0+1}^{n} = \{0\}$.

Suppose that $v + V_0$ is an arbitrary vector of V/V_0. There exists a subset $\{\nu_k\}_{k=1}^{n}$ of $\mathbb{F}$ such that
$$v = \sum_{k=1}^{n} \nu_k e_k$$
and we have
$$v + V_0 = \sum_{k=n_0+1}^{n} \nu_k (e_k + V_0).$$

2. Suppose that $\{e_k\}_{k=1}^{n_0}$ is a basis of V_0 and $\{e_k + V_0\}_{k=n_0+1}^{n}$ is a basis of V/V_0.

   Suppose that $\{\nu_k\}_{k=1}^{n}$ is a subset of $\mathbb{F}$ such that
$$\sum_{k=1}^{n} \nu_k e_k = 0.$$
   We have
$$\sum_{k=n_0+1}^{n} \nu_k (e_k + V_0) = 0$$
   and $\{\nu_k\}_{k=n_0+1}^{n} = \{0\}$. We have
$$\sum_{k=1}^{n_0} \nu_k e_k = 0$$
   and $\{\nu_k\}_{k=1}^{n_0} = \{0\}$.

   Suppose that v is an arbitrary vector of V. There exists a subset $\{\nu_k\}_{k=n_0+1}^{n}$ of $\mathbb{F}$ such that
$$v + V_0 = \sum_{k=n_0+1}^{n} \nu_k (e_k + V_0).$$
   There exists a subset $\{\nu_k\}_{k=1}^{n_0}$ of $\mathbb{F}$ such that
$$v - \sum_{k=n_0+1}^{n} \nu_k e_k = \sum_{k=1}^{n_0} \nu_k e_k$$
   and we have
$$v = \sum_{k=1}^{n} \nu_k e_k. \quad \square$$

DEFINITION 2.1.2
Suppose that x is a linear mapping on a vector space V and let V_0 be an invariant subspace for x.

1. We define a linear mapping x_{V_0} on V_0 by
$$x_{V_0}(v_0) = x(v_0).$$

2. We define a linear mapping x_{V/V_0} on V/V_0 by
$$x_{V/V_0}(v + V_0) = x(v) + V_0.$$

PROPOSITION 2.1.4
Suppose that x is a linear mapping on a vector space V and let V_0 be an invariant subspace for x.
$$V = V_0 \oplus V/V_0, \qquad\qquad x = \begin{pmatrix} x_{V_0} & * \\ 0 & x_{V/V_0} \end{pmatrix}.$$

PROPOSITION 2.1.5
Suppose that V_0 is a subspace of a vector space V.

1. The set
$$\left\{ x \in \hom V : V_0 \text{ is an invariant subspace for } x \right\} \tag{2.1}$$
is a subalgebra of the algebra $\hom V$.

2. The mapping
$$\left\{ x \in \hom V : V_0 \text{ is an invariant subspace for } x \right\} \to \hom V_0$$
is a homomorphism of algebras.

3. The mapping
$$\left\{ x \in \hom V : V_0 \text{ is an invariant subspace for } x \right\} \to \hom V/V_0$$
is a homomorphism of algebras.

COROLLARY 2.1.1
Suppose that ρ is a representation of a Lie algebra L on a vector space V and let V_0 be an invariant subspace for $\rho(L)$.

1. The mapping $x \mapsto \rho(x)$ is a homomorphism of L into the Lie algebra (2.1).

2. The mapping
$$x \mapsto \rho_{V_0}(x) = \left[v \mapsto \rho(x)v\right]$$
is a representation of L on V_0.

3. The mapping

$$x \mapsto \rho_{V/V_0}(x) = \big[v + V_0 \mapsto \rho(x)v + V_0\big]$$

is a representation of L on V/V_0.

4. $$\rho(x) = \begin{pmatrix} \rho_{V_0}(x) & * \\ 0 & \rho_{V/V_0}(x) \end{pmatrix}$$

for $\forall x$.

Corollary 2.1.2
Suppose that L_0 is a Lie subalgebra of a Lie algebra L.

$$x_0 \mapsto \operatorname{ad}_{L/L_0} x_0 = \big[x + L_0 \mapsto (\operatorname{ad} x_0)(x) + L_0\big]$$

is a representation of L_0 on L/L_0.

Proposition 2.1.6
Suppose that L_0 is a Lie subalgebra of a Lie algebra L.

Proposition 2.1.7
$\ker x \neq \{0\}$ if x is a nilpotent linear mapping on a vector space $V \neq \{0\}$.

Proof. We may assume that $x \neq 0$. We define $n = \min\{\, n : x^n = 0 \,\} > 1$. There exists a vector v such that $x^{n-1}v \neq 0$. We have $x(x^{n-1}v) = x^n v = 0$. □

Proposition 2.1.8
$\operatorname{ad} x$ is nilpotent if x is a nilpotent linear mapping on a vector space.

Proof. We define $n = \min\{\, n : x^n = 0 \,\}$.

$$(\operatorname{ad} x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} L(x)^k (-1)^{2n-k} R(x)^{2n-k} = 0. \quad \square$$

Theorem 2.1.1
Suppose that $V \neq \{0\}$ is a vector space over a field $\mathbb{F}$ and let L be a finite dimensional Lie subalgebra of $\hom V$. Assume that x is nilpotent for $\forall x$ of L.

$$\bigcap_{x \in L} \ker x \neq \{0\}.$$

Proof. Suppose that $\dim L = 0$.

$$\bigcap_{x \in L} \ker x = V \neq \{0\}.$$

Suppose that $\dim L > 0$ and let $\mathcal{L}^{(n)}$ be the set

$$\big\{\text{a Lie subalgebra of } L \text{ of dimension } n\big\}$$

for $\forall n$. We remark that the set $\mathcal{L}^{(0)} = \{\{0\}\}$ is not empty. We define

$$n = \max\{\, n \in \{0, \ldots, \dim L - 1\} : \text{the set } \mathcal{L}^{(n)} \text{ is not empty} \,\}$$

and let L_n be an element of $\mathcal{L}^{(n)}$. We have

$$V_n = \bigcap_{x_n \in L_n} \ker x_n \neq \{0\}$$

by the induction hypothesis. We remark that the linear mapping $\operatorname{ad}_{L/L_n} x_n$ is nilpotent for $\forall x_n$ of L_n by Proposition 2.1.8. We have

$$\bigcap_{x_n \in L_n} \ker(\operatorname{ad}_{L/L_n} x_n) \neq \{0\}$$

by the induction hypothesis.

$$x + L_n \in \bigcap_{x_n \in L_n} \ker(\operatorname{ad}_{L/L_n} x_n) \setminus \{0\} \Leftrightarrow x \in \bigcap_{x_n \in L_n} (\operatorname{ad}_L x_n)^{-1}(L_n) \setminus L_n.$$

Suppose that x is an element of

$$\bigcap_{x_n \in L_n} (\operatorname{ad}_L x_n)^{-1}(L_n) \setminus L_n.$$

The set $L_n + \mathbb{F}x$ belongs to $\mathcal{L}^{(n+1)}$. We have $L = L_n + \mathbb{F}x$.

$$x_n x V_n = \big((\operatorname{ad} x_n)(x) + x x_n\big) V_n = \{0\}$$

for $\forall x_n$ of L_n. The subspace V_n is invariant for x.

$$\bigcap_{x \in L} \ker x = \ker x \cap V_n \neq \{0\}$$

by Proposition 2.1.7. □

Engel's Theorem

Suppose that V is a finite dimensional vector space over a field $\mathbb{F}$ and let f be a representation of a Lie algebra L on V such that each $f(x)$ is nilpotent. There exists a basis such that the matrix representation of each $f(x)$ is strictly upper triangular.

Proof. The proof is by induction on $\dim V$. We may assume that $\dim V > 0$.

$$\bigcap_{x \in L} \ker f(x) \neq \{0\}$$

by Theorem 2.1.1 and let e_1 be an element of

$$\bigcap_{x \in L} \ker f(x) \setminus \{0\}.$$

We have $f(L)(e_1) = \{0\}$. A subspace $\mathbb{F}e_1$ is invariant for f.

$$f(x) = \begin{pmatrix} 0 & * \\ 0 & f_{V/\mathbb{F}e_1}(x) \end{pmatrix}$$

for $\forall x$ and each $f_{V/\mathbb{F}e_1}(x)$ is nilpotent. □

Corollary 2.1.3
A finite dimensional Lie algebra is nilpotent if and only if each $\operatorname{ad} x$ is nilpotent.

Proof. Suppose that each $\operatorname{ad} x$ is nilpotent. There exists a basis such that the matrix representation of each $\operatorname{ad} x$ is strictly upper triangular by Engel's theorem. We write n for the dimension of the Lie algebra. We have

$$\left\{ (\operatorname{ad} x_1) \cdots (\operatorname{ad} x_n) : (x_k)_{k=1}^n \right\} = \{0\}. \quad \square$$

Corollary 2.1.4
Suppose that V is a finite dimensional vector space. A Lie subalgebra L of $\hom V$ is nilpotent if each element of L is a nilpotent linear mapping on V.

Proof. Suppose that x is an element of L. The element $\operatorname{ad}_{\hom V} x$ is a nilpotent linear mapping on $\hom V$ by Proposition 2.1.8 and the subspace L is invariant for $\operatorname{ad}_{\hom V} x$. The element $\operatorname{ad}_L x$ is a nilpotent linear mapping on L. The finite dimensional Lie algebra L is nilpotent by Corollary 2.1.3. □

Corollary 2.1.5
Suppose that $\mathbb{F}$ is a field. The set of strictly upper triangular matrices

$$\left\{ x \in M(m, \mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \right\}$$

is a nilpotent Lie subalgebra of $M(m, \mathbb{F})$ for $\forall m$.

Proof. By Proposition 1.2.8 and Corollary 2.1.4. □

Corollary 2.1.6
Suppose that L is a finite dimensional nilpotent Lie algebra. There exists a basis of L such that the matrix representation of each $\operatorname{ad} x$ is strictly upper triangular.

Corollary 2.1.7
The Killing form of a finite dimensional nilpotent Lie algebra is trivial.

Corollary 2.1.8
Suppose that V is a finite dimensional vector space over a field $\mathbb{F}$ and let x be a nilpotent element of $\hom V$. There exists a basis such that the matrix representation of x is strictly upper triangular.

Proof. The subspace $\mathbb{F}x$ is a Lie subalgebra and by Engel's theorem. □

2.2 Lie's Theorem

Definition 2.2.1
A Lie algebra L is said to be solvable if eventually

$$L, [L, L], \big[[L, L], [L, L]\big], \ldots, \{0\}, \ldots.$$

Proposition 2.2.1
A nilpotent Lie algebra is solvable.

Suppose that ρ is a representation of a Lie algebra L on a vector space V. Suppose that L_0 is an ideal of L and let f_0 be a linear functional on L_0. We define

$$V_{f_0} = \big\{\, v : \rho(x_0)v = f_0(x_0)v \text{ for } \forall x_0 \,\big\}.$$

Suppose that (x, v) is an element of $L \times V_{f_0}$.

Proposition 2.2.2

$$\rho(x_0)\rho(x)^n v = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_0\big((\operatorname{ad} x)^k x_0\big)\rho(x)^{n-k} v$$

for $\forall (x_0, n)$ of $L_0 \times \mathbb{Z}^+$.

Proof.

$$\begin{aligned}
\rho(x_0)\rho(x)^n v &= \Big(\big[\rho(x_0), \rho(x)\big] + \rho(x)\rho(x_0)\Big)\rho(x)^{n-1}(x)v \\
&= \rho\big([x_0, x]\big)\rho(x)^{n-1}v + \rho(x)\rho(x_0)\rho(x)^{n-1}(x)v \\
&= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\Big(f_0\big((\operatorname{ad} x)^k[x_0, x]\big) + \rho(x)f_0\big((\operatorname{ad} x)^k x_0\big)\Big)\rho(x)^{n-1-k}v \\
&= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\Big(-f_0\big((\operatorname{ad} x)^{k+1} x_0\big) + \rho(x)f_0\big((\operatorname{ad} x)^k x_0\big)\Big)\rho(x)^{n-1-k}v \\
&= \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1} f_0\big((\operatorname{ad} x)^k x_0\big)\rho(x)^{n-k}v \\
&\quad + \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k} f_0\big((\operatorname{ad} x)^k x_0\big)\rho(x)^{n-k}v \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k} f_0\big((\operatorname{ad} x)^k x_0\big)\rho(x)^{n-k}v. \quad \square
\end{aligned}$$

Definition 2.2.2
We write U_n for the subspace generated by $\big\{v, \rho(x)v, \ldots, \rho(x)^{n-1}v\big\}$ for $\forall n$.

Proposition 2.2.3
Suppose that $\dim V < \infty$ and let $n = \min\{\, n : U_n = U_{n+1} \,\}$. We have $nf_0\big((\operatorname{ad} x)x_0\big) = 0$ for $\forall x_0$ of L_0.

Proof. We have $\dim U_n = n$.

$$\begin{aligned}\rho(x_0)\bigl(v, \rho(x)v, \ldots, \rho(x)^{n-1}v\bigr) &= \bigl(v, \rho(x)v, \ldots, \rho(x)^{n-1}v\bigr) \\ &\sum_{1\le i\le j\le n} (-1)^{j-i}\binom{j-1}{j-i} f_0\bigl((\operatorname{ad} x)^{j-i}x_0\bigr)e_{ij} \\ &= \bigl(v, \rho(x)v, \ldots, \rho(x)^{n-1}v\bigr) \\ &\Bigl(f_0(x_0) + \sum_{1\le i<j\le n} (-1)^{j-i}\binom{j-1}{j-i} f_0\bigl((\operatorname{ad} x)^{j-i}x_0\bigr)e_{ij}\Bigr) \qquad (2.2)\end{aligned}$$

by Proposition 2.2.2. We define $A(x)$ by

$$\rho(x)\bigl(v, \rho(x)v, \ldots, \rho(x)^{n-1}v\bigr) = \bigl(v, \rho(x)v, \ldots, \rho(x)^{n-1}v\bigr)A(x)$$

and $A(x_0, x)$ by

$$\rho(x_0)\bigl(v, \rho(x)v, \ldots, \rho(x)^{n-1}v\bigr) = \bigl(v, \rho(x)v, \ldots, \rho(x)^{n-1}v\bigr)A(x_0, x).$$

$$\begin{aligned}n f_0\bigl((\operatorname{ad} x)x_0\bigr) &= \operatorname{tr} A\bigl((\operatorname{ad} x)x_0, x\bigr) \\ &= \operatorname{tr}\Bigl(\bigl(\operatorname{ad} A(x)\bigr)A(x_0, x)\Bigr) = 0\end{aligned}$$

by the equation (2.2). □

PROPOSITION 2.2.4
Suppose that $\dim V < \infty$ and the underlying field is of characteristic 0.

$$V_{f_0} = \{\, v : \rho(x_0)v = f_0(x_0)v \text{ for } \forall x_0 \,\}$$

is an invariant subspace for ρ.

Proof. We may assume that $V_{f_0} \neq \{0\}$. Suppose that (x, v) is an element of $L \times V_{f_0} \setminus \{0\}$.

$$n = \min\{\, n : U_n = U_{n+1} \,\} \geq 1.$$

Suppose that (x, x_0) is an arbitrary element of $L \times L_0$.

$$f_0\bigl((\operatorname{ad} x)x_0\bigr) = 0$$

by Proposition 2.2.3.

$$\begin{aligned}\rho(x_0)\rho(x)v &= f_0(x_0)\rho(x)v - f_0\bigl((\operatorname{ad} x)x_0\bigr)v \\ &= f_0(x_0)\rho(x)v\end{aligned}$$

by Proposition 2.2.2 and the subspace V_{f_0} is invariant for ρ. □

Theorem 2.2.1
Suppose that ρ is a representation of a finite dimensional solvable Lie algebra over an algebraically closed field of characteristic 0 on a finite dimensional vector space $V \neq \{0\}$. There exists a linear functional f such that

$$V_f = \big\{\, v : \rho(x)v = f(x)v \text{ for } \forall x \,\big\} \neq \{0\}.$$

Proof. We write n for the dimension of the Lie algebra L and the proof is by induction on n. Suppose that $n > 0$. There exists a subspace L_0 of dimension $n-1$ containing $[L, L]$. The subspace L_0 is a solvable ideal. There exists a linear functional f_0 on L_0 such that

$$V_{f_0} = \big\{\, v : \rho(x_0)v = f_0(x_0)v \text{ for } \forall x_0 \,\big\} \neq \{0\}$$

by the induction hypothesis. The subspace V_{f_0} is invariant for ρ by Proposition 2.2.4.

Suppose that x is an element of $L \setminus L_0$. The linear mapping $\rho_{V_{f_0}}(x)$ has an eigenvalue ν since the underlying field is algebraically closed. Suppose that v is an element of $\ker\big(\nu - \rho_{V_{f_0}}(x)\big) \setminus \{0\}$. There exists a unique linear functional f extending f_0 such that $f(x) = \nu$. The vector v belongs to V_f. □

Lie's Theorem
Suppose that ρ is a representation of a solvable Lie algebra over an algebraically closed field of characteristic 0 on a finite dimensional vector space. There exists a basis such that the matrix representation of each $\rho(x)$ is upper triangular.

Proof. We write n for the dimension of the vector space V and the proof is by induction on n. Suppose that $n > 0$. There exists a linear functional f such that

$$V_f = \big\{\, v : \rho(x)v = f(x)v \text{ for } \forall x \,\big\} \neq \{0\}$$

by Theorem 2.2.1.

We write $\mathbb{F}$ for the underlying field and let e_1 be a vector of $V_f \setminus \{0\}$. The subspace $\mathbb{F}e_1$ is invariant for ρ. There exists a basis $\big\{e_k + \mathbb{F}e_1\big\}_{k=2}^{n}$ such that the matrix representation of each $\rho_{V/\mathbb{F}e_1}(x)$ is upper triangular by the induction hypothesis.

$$\rho(x) = \begin{pmatrix} f(x) & * \\ 0 & \rho_{V/\mathbb{F}e_1}(x) \end{pmatrix}$$

for $\forall x$. □

Proposition 2.2.5
Suppose that R_1 is a commutative ring with ideneity and let R_2 be a subring with identity of R_1. The ring R_1 is a commutative algebra with identity over R_2.

Proof. The ring R_1 is a unital module over R_2 such that $(\nu_2\nu_{11})\nu_{12} = \nu_2(\nu_{11}\nu_{12}) = \nu_{11}(\nu_2\nu_{12})$ for $\forall(\nu_{11}, \nu_{12}, \nu_2)$ of $R_1^2 \times R_2$. □

Suppose that R_2 is a commutative ring with identity and let R_1 be a commutative algebra with identity over R_2.

Proposition 2.2.6
A unital module over R_1 is a unital module over R_2.

Proof. Suppose that X is a unital module over R_1.

Suppose that (ν_2, x_1, x_2) is an arbitrary element of $R_2 \times X^2$. We have

$$\begin{aligned}\nu_2(x_1 + x_2) &= (\nu_2 1_{R_1})(x_1 + x_2)\\ &= (\nu_2 1_{R_1})x_1 + (\nu_2 1_{R_1})x_2\\ &= \nu_2 x_1 + \nu_2 x_2.\end{aligned}$$

Suppose that (ν_{21}, ν_{22}, x) is an arbitrary element of $R_2^2 \times X$. We have

$$\begin{aligned}(\nu_{21} + \nu_{22})x &= \big((\nu_{21} + \nu_{22})1_{R_1}\big)x\\ &= (\nu_{21}1_{R_1} + \nu_{22}1_{R_1})x\\ &= (\nu_{21}1_{R_1})x + (\nu_{22}1_{R_1})x\\ &= \nu_{21}x + \nu_{22}x\end{aligned}$$

and we have

$$\begin{aligned}(\nu_{21}\nu_{22})x &= \big((\nu_{21}\nu_{22})1_{R_1}\big)x\\ &= \big((\nu_{21}1_{R_1})(\nu_{22}1_{R_1})\big)x\\ &= (\nu_{21}1_{R_1})\big((\nu_{22}1_{R_1})x\big)\\ &= \nu_{21}(\nu_{22}x).\end{aligned}$$

Suppose that x is an arbitrary element of X. We have

$$\begin{aligned}1_{R_2}x &= (1_{R_2}1_{R_1})x\\ &= 1_{R_1}x\\ &= x.\end{aligned}$$ □

Suppose that X_1 is a unital module over R_1 and let X_2 be a unital module over R_2.

Proposition 2.2.7
The tensor product $X_1 \otimes X_2$ is a compatible unital module over R_1.

Proof. Suppose that ν_1 is an element of R_1. A mapping

$$X_1 \times X_2 \to X_1 \otimes X_2, \qquad (x_1, x_2) \mapsto (\nu_1 x_1) \otimes x_2$$

is bilinear over R_2 since

$$\big(\nu_1(\nu_2 x_1)\big) \otimes x_2 = \big(\nu_2(\nu_1 x_1)\big) \otimes x_2 = \nu_2\big((\nu_1 x_1) \otimes x_2\big)$$

for $\forall \nu_2$ of R_2.

$$x_1 \otimes x_2 \mapsto (\nu_1 x_1) \otimes x_2$$

defines a unique homomorphism over R_2 on $X_1 \otimes X_2$. We define

$$\nu_1 \sum_{k=1}^{n} x_{1k} \otimes x_{2k} = \sum_{k=1}^{n} (\nu_1 x_{1k}) \otimes x_{2k}$$

for $\forall \nu_1$ of R_1 and for

$$\forall x = \sum_{k=1}^{n} x_{1k} \otimes x_{2k}$$

of $X_1 \otimes X_2$. Suppose that

$$(\nu_{11}, \nu_{12}, x = \sum_{k=1}^{n} x_{1k} \otimes x_{2k})$$

is an arbitrary element of $R_1^2 \times (X_1 \otimes X_2)$. We have

$$\begin{aligned}
(\nu_{11} + \nu_{12})x &= \sum_{k=1}^{n} \big((\nu_{11} + \nu_{12})x_{1k}\big) \otimes x_{2k} \\
&= \sum_{k=1}^{n} (\nu_{11} x_{1k}) \otimes x_{2k} + \sum_{k=1}^{n} (\nu_{12} x_{1k}) \otimes x_{2k} \\
&= \nu_{11} x + \nu_{12} x
\end{aligned}$$

and we have

$$\begin{aligned}
(\nu_{11}\nu_{12})x &= \sum_{k=1}^{n} \big((\nu_{11}\nu_{12})x_{1k}\big) \otimes x_{2k} \\
&= \nu_{11} \sum_{k=1}^{n} (\nu_{12} x_{1k}) \otimes x_{2k} \\
&= \nu_{11}(\nu_{12} x).
\end{aligned}$$

We have

$$1_{R_1} x = \sum_{k=1}^{n} (1_{R_1} x_{1k}) \otimes x_{2k} = x.$$

The tensor product $X_1 \otimes X_2$ is a unital module over R_1.

Suppose that ν_2 is an arbitrary element of R_2. We have

$$\nu_2 x = \sum_{k=1}^{n} (\nu_2 x_{1k}) \otimes x_{2k} = \sum_{k=1}^{n} \big((\nu_2 1_{R_1}) x_{1k}\big) \otimes x_{2k} = (\nu_2 1_{R_1}) x. \quad \square$$

Proposition 2.2.8

Suppose that X and Y are unital modules over R_1. A homomorphism over R_1 of X into Y is a homomorphism over R_2.

Proof. Suppose that f is a homomorphism over R_1 of X into Y and let (ν_2, x) be an arbitrary element of $R_2 \times X$. We have

$$\begin{aligned} f(\nu_2 x) &= f\big((\nu_2 1_{R_1})x\big) \\ &= (\nu_2 1_{R_1}) f(x) \\ &= \nu_2 f(x). \end{aligned}$$ □

Proposition 2.2.9
Suppose that R is a commutative ring with identity and let $(X_k)_{k=1}^n$ be a finite sequence of unital modules over R. Suppose that Y is a unital module over R. The unital module $\hom(\bigotimes_{k=1}^n X_k, Y)$ is the set of multilinear mapping of $\bigoplus_{k=1}^n X_k$ into Y.

$$\begin{tikzcd} \bigoplus_{k=1}^n X_k \arrow[r] \arrow[d] & Y \\ \bigotimes_{k=1}^n X_k \arrow[ur] & \end{tikzcd}$$

Theorem 2.2.2
We have $\hom_{R_1}(R_1 \otimes X_2, X_1) = \hom_{R_2}(X_2, X_1)$.

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_1 \\ R_1 \otimes X_2 \arrow[ur] & \end{tikzcd}$$

Proof. By Theorem of 487 (cf. Algebra). □

Suppose that Y_1 is a unital module over R_1 and let Y_2 be a unital module over R_2.

Corollary 2.2.1
The mapping

$$\hom_{R_2}(X_2, Y_2) \to \hom_{R_1}(R_1 \otimes X_2, R_1 \otimes Y_2), \qquad f \mapsto \big[1 \otimes x \mapsto 1 \otimes f(x)\big]$$

is a homomorphism over R_2.

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & Y_2 \arrow[d] \\ R_1 \otimes X_2 \arrow[r] & R_1 \otimes Y_2 \end{tikzcd}$$

Corollary 2.2.2
Suppose that Y is a unital module over R_1 and let $(X_k)_{k=1}^n$ be a finite sequence of unital modules over R_2. We have

$$\hom_{R_1}(\bigotimes_{k=1}^n R_1 \otimes X_k, Y) = \hom_{R_2}(\bigotimes_{k=1}^n X_k, Y).$$

$$\begin{tikzcd}
\bigotimes_{k=1}^n X_k \arrow[r] \arrow[d] & Y \\
\bigotimes_{k=1}^n R_1 \otimes X_k \arrow[ur] &
\end{tikzcd}$$

Proof. We remark that we have

$$R_1 \otimes \bigotimes_{k=1}^n X_k = \bigotimes_{k=1}^n R_1 \otimes X_k$$

by Proposition of 425 (cf. Algebra). □

Corollary 2.2.3

$$\begin{tikzcd}
 & & \bigotimes_{k=1}^n X_k \arrow[dd] \arrow[dddl] \\
\bigoplus_{k=1}^n X_k \arrow[urr] \arrow[dd] \arrow[ddr] & & \\
 & & \bigotimes_{k=1}^n X_k \otimes R_1 \arrow[dl] \\
\bigoplus_{k=1}^n X_k \otimes R_1 \arrow[r] \arrow[urr] & Y &
\end{tikzcd}$$

Proof. We remark that we have

$$R_1 \otimes \bigoplus_{k=1}^n X_k = \bigoplus_{k=1}^n R_1 \otimes X_k$$

by Proposition of 436 (cf. Algebra). Suppose that f is a multilinear mapping of $\bigoplus_{k=1}^n X_k$ into Y. We have

$$f(\bigoplus_{k=1}^n x_k \otimes 1) = f(\bigoplus_{k=1}^n x_k). \quad \square$$

Corollary 2.2.4

Suppose that Y is a unital module over R_2. The mapping

$$\hom_{R_2}(\bigotimes_{k=1}^n X_k, Y) \to \hom_{R_1}(\bigotimes_{k=1}^n X_k \otimes R_1, Y \otimes R_1),$$

$$f \mapsto \left[\bigotimes_{k=1}^n x_k \otimes 1 \mapsto f(\bigotimes_{k=1}^n x_k) \otimes 1\right]$$

is a homomorphism over R_2.

$$\begin{tikzcd}
\bigotimes_{k=1}^n X_k \arrow[r] \arrow[d] & Y \arrow[d] \\
\bigotimes_{k=1}^n X_k \otimes R_1 \arrow[r] & Y \otimes R_1
\end{tikzcd}$$

Corollary 2.2.5

$$\begin{tikzcd}
 & & \bigotimes_{k=1}^n X_k \arrow[dl] \arrow[dd] \\
\bigoplus_{k=1}^n X_k \arrow[urr] \arrow[r] \arrow[dd] & Y \arrow[dd] & \\
 & & \bigotimes_{k=1}^n X_k \otimes R_1 \arrow[dl] \\
\bigoplus_{k=1}^n X_k \otimes R_1 \arrow[urr] \arrow[r] & Y \otimes R_1 &
\end{tikzcd}$$

Proposition 2.2.10

Suppose that f_1 is a homomorphism over R_1 of X_1 into Y_1 and let f_2 be a homomorphism over R_2 of X_2 into Y_2. A homomorphism over R_2

$$X = X_1 \otimes X_2 \to Y = Y_1 \otimes Y_2, \qquad x \mapsto f(x) = (f_1 \otimes f_2)(x)$$

is a homomorphism over R_1.

Proof. We have

$$\begin{aligned}
f(\nu_1 x) &= f\Big(\sum_{k=1}^n (\nu_1 x_{1k}) \otimes x_{2k}\Big) \\
&= \sum_{k=1}^n f_1(\nu_1 x_{1k}) \otimes f_2(x_{2k}) \\
&= \sum_{k=1}^n \big(\nu_1 f_1(x_{1k})\big) \otimes f_2(x_{2k}) \\
&= \nu_1 \sum_{k=1}^n f_1(x_{1k}) \otimes f_2(x_{2k}) \\
&= \nu_1 f(x)
\end{aligned}$$

for $\forall(\nu_1, x = \sum_{k=1}^n x_{1k} \otimes x_{2k})$ of $R_1 \times X$. $\square$

Suppose that V is a vector space over a field $\mathbb{F}$ and let $\mathbb{E}$ be an extension field of $\mathbb{F}$.

Proposition 2.2.11

We have the following.

1. The set V is a subspace over $\mathbb{F}$ of $\mathbb{E} \otimes V$ and we have

$$\mathbb{E} \otimes V = \langle V \rangle_{\mathbb{E}}.$$

2. A basis of V over $\mathbb{F}$ is a basis of $\mathbb{E} \otimes V$ over $\mathbb{E}$ and we have

$$\dim_{\mathbb{E}}(\mathbb{E} \otimes V) = \dim_{\mathbb{F}} V.$$

Proof. 1. The vector space $V = \mathbb{F} \otimes V$ is a subspace of $\mathbb{E} \otimes V$ since $\mathbb{F}$ is a subspace of $\mathbb{E}$.

2. Suppose that Λ is a basis of V. We have

$$\begin{aligned} \mathbb{E} \otimes V &= \mathbb{E} \otimes \mathbb{F}^{\oplus \Lambda} \\ &= (\mathbb{E} \otimes \mathbb{F})^{\oplus \Lambda} \\ &= \mathbb{E}^{\oplus \Lambda} \end{aligned}$$

by Proposition of 436 (cf. Algebra). $\square$

Proposition 2.2.12

A Lie algebra over R_1 is a Lie algebra over R_2.

Suppose that X_2 is a Lie algebra over R_2. We define the bilinear mapping

$$\Big[\sum_{k_1=1}^{n_1} \nu_{1k_1} \otimes x_{1k_1}, \sum_{k_2=1}^{n_2} \nu_{2k_2} \otimes x_{2k_2}\Big] = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \nu_{1k_1}\nu_{2k_2} \otimes \big[x_{1k_1}, x_{2k_2}\big] \tag{2.3}$$

for $\forall (x_1, x_2) = (\sum_{k_1=1}^{n_1} \nu_{1k_1} \otimes x_{1k_1}, \sum_{k_2=1}^{n_2} \nu_{2k_2} \otimes x_{2k_2})$ of $R_1 \otimes X_2^{\oplus 2}$ by Corollary 2.2.5.

$$\begin{tikzcd}
 & & X_2^{\otimes 2} \arrow[dl] \arrow[dd] \\
X_2^{\otimes 2} \arrow[urr] \arrow[r] \arrow[dd] & X_2 \arrow[dd] & \\
 & & R_1 \otimes X_2^{\otimes 2} \arrow[dl] \\
R_1 \otimes X_2^{\oplus 2} \arrow[urr] \arrow[r] & R_1 \otimes X_2 &
\end{tikzcd}$$

Proposition 2.2.13

The bilinear mapping (2.3) satisfies the Jacobi identity.

Proof. Suppose that

$$(x_1, x_2, x_3) = (\sum_{k_1=1}^{n_1} \nu_{1k_1} \otimes x_{1k_1}, \sum_{k_2=1}^{n_2} \nu_{2k_2} \otimes x_{1k_2}, \sum_{k_3=1}^{n_3} \nu_{3k_3} \otimes x_{3k_3})$$

is an arbitrary element of $R_1 \otimes X_2^{\oplus 3}$. We have

$$\big[x_1, [x_2, x_3]\big] + \big[x_2, [x_3, x_1]\big] + \big[x_3, [x_1, x_2]\big] = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{k_3=1}^{n_3} \nu_{1k_1}\nu_{2k_2}\nu_{3k_3}$$
$$\otimes \Big(\big[x_{1k_1}, [x_{2k_2}, x_{3k_3}]\big] + \big[x_{2k_2}, [x_{3k_3}, x_{1k_1}]\big] + \big[x_{3k_3}, [x_{1k_1}, x_{2k_2}]\big]\Big) = 0. \quad \square$$

THEOREM 2.2.3

The unital module $R_1 \otimes X_2$ is a Lie algebra.

Proof. Suppose that $x = \sum_{k=1}^{n} \nu_k \otimes x_k$ is an arbitrary element of $R_1 \otimes X_2$. We have

$$[x, x] = \sum_{i,j=1}^{n} \nu_i \nu_j \otimes [x_i, x_j] = 0. \quad \square$$

Suppose that X_1 is a Lie algebra over R_1.

PROPOSITION 2.2.14

Suppose that $R_1 \otimes X_2$ is a Lie algebra. We have the following.

1. The mapping

$$X_2 \to R_1 \otimes X_2, \qquad x \mapsto 1 \otimes x \tag{2.4}$$

is a homomorphism of Lie algebras.

2. We have $\hom_{R_1}(R_1 \otimes X_2, X_1) = \hom_{R_2}(X_2, X_1)$.

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_1 \\ R_1 \otimes X_2 \arrow[ur] & \end{tikzcd}$$

Proof. The set $\hom_{R_1}(R_1 \otimes X_2, X_1)$ is contained in $\hom_{R_2}(X_2, X_1)$ since the mapping (2.4) is a homomorphism of Lie algebras. Suppose that f is an element of $\hom_{R_2}(X_2, X_1)$. We have

$$\begin{aligned} f\big([x_1, x_2]\big) &= \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \nu_{1k_1}\nu_{2k_2} f\big([x_{1k_1}, x_{2k_2}]\big) \\ &= \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \nu_{1k_1}\nu_{2k_2} \big[f(x_{1k_1}), f(x_{2k_2})\big] \\ &= \big[f(\sum_{k_1=1}^{n_1} \nu_{1k_1} \otimes x_{1k_1}), f(\sum_{k_2=1}^{n_2} \nu_{2k_2} \otimes x_{2k_2})\big] \\ &= \big[f(x_1), f(x_2)\big] \end{aligned}$$

for $\forall (x_1, x_2) = (\sum_{k_1=1}^{n_1} \nu_{1k_1} \otimes x_{1k_1}, \sum_{k_2=1}^{n_2} \nu_{2k_2} \otimes x_{2k_2})$ of $R_1 \otimes X_2^{\oplus 2}$. □

PROPOSITION 2.2.15
An algebra over a commutative ring with identity is a Lie algebra.

PROPOSITION 2.2.16
An algebra over R_1 is an algebra over R_2.

Proof. By Proposition 2.2.6. □

Suppose that X_1 is an algebra over R_1 and let X_2 be an algebra over R_2.

PROPOSITION 2.2.17
The tensor product $X_1 \otimes X_2$ is a compatible algebra over R_1.

COROLLARY 2.2.6
The module $R_1 \otimes X_2$ is a Lie algebra.

Proof. We have

$$\begin{aligned}[x_1, x_2] &= \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \nu_{1k_1} \nu_{2k_2} \otimes \left[x_{1k_1}, x_{2k_2}\right] \\ &= \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \nu_{1k_1} \nu_{2k_2} \otimes (x_{1k_1} x_{2k_2} - x_{2k_2} x_{1k_1}) \\ &= x_1 x_2 - x_2 x_1\end{aligned}$$

for $\forall (x_1, x_2) = (\sum_{k_1=1}^{n_1} \nu_{1k_1} \otimes x_{1k_1}, \sum_{k_2=1}^{n_2} \nu_{2k_2} \otimes x_{2k_2})$ of $R_1 \otimes X_2^{\oplus 2}$. □

PROPOSITION 2.2.18
The mapping

$$X_2 \to R_1 \otimes X_2, \qquad x \mapsto 1 \otimes x \tag{2.5}$$

is a homomorphism of algebras.

THEOREM 2.2.4
We have $\hom_{R_1}(R_1 \otimes X_2, X_1) = \hom_{R_2}(X_2, X_1)$.

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_1 \\ R_1 \otimes X_2 \arrow[ur] & \end{tikzcd}$$

Proof. The set $\hom_{R_1}(R_1 \otimes X_2, X_1)$ is contained in $\hom_{R_2}(X_2, X_1)$ since the mapping (2.5) is a homomorphism of algebras. Suppose that f is an element of

$\hom_{R_2}(X_2, X_1)$. We have

$$\begin{aligned}
f(x_1x_2) &= \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \nu_{1k_1}\nu_{2k_2} f(x_{1k_1}x_{2k_2}) \\
&= \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \nu_{1k_1}\nu_{2k_2} f(x_{1k_1}) f(x_{2k_2}) \\
&= f(\sum_{k_1=1}^{n_1} \nu_{1k_1} \otimes x_{1k_1}) f(\sum_{k_2=1}^{n_2} \nu_{2k_2} \otimes x_{2k_2}) \\
&= f(x_1)f(x_2)
\end{aligned}$$

for $\forall (x_1, x_2) = (\sum_{k_1=1}^{n_1} \nu_{1k_1} \otimes x_{1k_1}, \sum_{k_2=1}^{n_2} \nu_{2k_2} \otimes x_{2k_2})$ of $R_1 \otimes X_2^{\oplus 2}$. □

Proposition 2.2.19

The algebra $\hom V$ is a subalgebra of $\hom(\mathbb{E} \otimes V)$.

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & V \arrow[d] \\
\mathbb{E} \otimes V \arrow[r] & \mathbb{E} \otimes V
\end{tikzcd}$$

Proof. An element of $\hom V$ extends uniquely to an element of $\hom(\mathbb{E} \otimes V)$ since $\hom(\mathbb{E} \otimes V) = \hom(V, \mathbb{E} \otimes V)$ by Theorem 2.2.2. □

Corollary 2.2.7

The algebra $M(n, \mathbb{F})$ is a subalgebra of $M(n, \mathbb{E})$ for $\forall n$.

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r] \arrow[d] & \mathbb{F}^n \arrow[d] \\
\mathbb{E}^n \arrow[r] & \mathbb{E}^n
\end{tikzcd}$$

There exists a unique homomorphism of $\mathbb{E} \otimes \hom V$ into $\hom(\mathbb{E} \otimes V)$ extending the identity mapping on $\hom V$ since

$$\hom\bigl(\mathbb{E} \otimes \hom V, \hom(\mathbb{E} \otimes V)\bigr) = \hom\bigl(\hom V, \hom(\mathbb{E} \otimes V)\bigr)$$

by Theorem 2.2.4.

Theorem 2.2.5

We have $\mathbb{E} \otimes \hom V = \hom(\mathbb{E} \otimes V)$ provided that V is finite dimensional.

Proof. The set of matrix units is a basis of

$$\hom(\mathbb{E} \otimes V) = M(n, \mathbb{E})$$

by Corollary 2.2.7.

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "e_{ij}"] \arrow[d] & \mathbb{F}^n \arrow[d] \\
\mathbb{E}^n \arrow[r, "e_{ij}"] & \mathbb{E}^n
\end{tikzcd}$$ □

Corollary 2.2.8

The following diagram commutes provided that V is finite dimensional.

$$\begin{tikzcd} \hom V \arrow[r, "\mathrm{tr}"] \arrow[d] & \mathbb{F} \arrow[d] \\ \hom(\mathbb{E} \otimes V) \arrow[r, "\mathrm{tr}"] & \mathbb{E} \end{tikzcd}$$

Suppose that L is a Lie algebra over $\mathbb{F}$.

Proposition 2.2.20

The set $\hom(L, \hom V)$ is contained in the set $\hom\big(\mathbb{E} \otimes L, \hom(\mathbb{E} \otimes V)\big)$.

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \hom V \arrow[d] \\ \mathbb{E} \otimes L \arrow[r] & \hom(\mathbb{E} \otimes V) \end{tikzcd}$$

Proposition 2.2.21

The following diagram commutes.

$$\begin{tikzcd} L \arrow[r, "\mathrm{ad}"] \arrow[d] & \hom L \arrow[d] \\ \mathbb{E} \otimes L \arrow[r, "\mathrm{ad}"] & \hom(\mathbb{E} \otimes L) \end{tikzcd}$$

Proposition 2.2.22

The following diagram commutes provided that L is finite dimensional.

$$\begin{tikzcd} L^{\oplus 2} \arrow[r, "B"] \arrow[d] & \mathbb{F} \arrow[d] \\ \mathbb{E} \otimes L^{\oplus 2} \arrow[r, "B"] & \mathbb{E} \end{tikzcd}$$

Proposition 2.2.23

Suppose that L_1 and L_2 are ideals of L. The subspace $\mathrm{span}[L_1, L_2]$ is an ideal of L.

Proof. We have

$$\big[x, [x_1, x_2]\big] = \big[[x, x_1], x_2\big] + \big[x_1, [x, x_2]\big]$$

and $\big[x, [x_1, x_2]\big]$ belongs to $\mathrm{span}[L_1, L_2]$ for $\forall (x, x_1, x_2)$ of $L \times L_1 \times L_2$. □

Definition 2.2.3

Suppose that S is a subset of L. We define a subspace $CS = \mathrm{span}[L, S]$.

Definition 2.2.4

The ideal $DL = \mathrm{span}[L, L]$ is called the derived Lie algebra.

Proposition 2.2.24
Suppose that S_1 and S_2 are subsets of L. We have

$$\operatorname{span}[S_1, S_2] = [\operatorname{span} S_1, \operatorname{span} S_2].$$

Proposition 2.2.25
We have

$$(C^n L)_{n=0}^{\infty} = \Big(\operatorname{span} L, \operatorname{span}[L, L], \operatorname{span}\big[L, [L, L]\big], \dots\Big).$$

Proof. The proof is by induction on n. Suppose that $n > 0$. We define

$$(S_n)_{n=0}^{\infty} = \Big(L, [L, L], \big[L, [L, L]\big], \dots\Big).$$

We have

$$\begin{aligned} C^n L &= \operatorname{span}\big[L, C^{n-1} L\big] \\ &= \operatorname{span}\big[L, \operatorname{span} S_{n-1}\big] \\ &= \operatorname{span}\big[L, S_{n-1}\big] \\ &= \operatorname{span} S_n. \end{aligned}$$

□

Corollary 2.2.9
A Lie algebra L is nilpotent if and only if $C^n L = \{0\}$ for some n.

Proposition 2.2.26
We have

$$(D^n L)_{n=0}^{\infty} = \Big(\operatorname{span} L, \operatorname{span}[L, L], \operatorname{span}\big[[L, L], [L, L]\big], \dots\Big).$$

Proof. The proof is by induction on n. Suppose that $n > 0$. We define

$$(S_n)_{n=0}^{\infty} = \Big(L, [L, L], \big[[L, L], [L, L]\big], \dots\Big).$$

We have

$$\begin{aligned} D^n L &= \operatorname{span}\big[D^{n-1} L, D^{n-1} L\big] \\ &= \operatorname{span}\big[\operatorname{span} S_{n-1}, \operatorname{span} S_{n-1}\big] \\ &= \operatorname{span}\big[S_{n-1}, S_{n-1}\big] \\ &= \operatorname{span} S_n. \end{aligned}$$

□

Corollary 2.2.10
A Lie algebra L is solvable if and only if $D^n L = \{0\}$ for some n.

Proposition 2.2.27
The sets $C^n L$ and $D^n L$ are ideals of L for $\forall n$.

Proof. By Proposition 2.2.23. □

Proposition 2.2.28
We have $C^n(\mathbb{E}\otimes L)=\mathbb{E}\otimes C^nL$ for $\forall n$.

Proof. The proof is by induction on n. Suppose that $n>0$. We have

$$\begin{aligned} C^n(\mathbb{E}\otimes L) &= \operatorname{span}\bigl[\mathbb{E}\otimes L, C^{n-1}(\mathbb{E}\otimes L)\bigr] \\ &= \operatorname{span}\bigl[\mathbb{E}\otimes L, \mathbb{E}\otimes C^{n-1}L\bigr] \\ &= \operatorname{span}\bigl[L, C^{n-1}L\bigr] \\ &= \operatorname{span} C^nL \\ &= \mathbb{E}\otimes C^nL \end{aligned}$$

by the induction hypothesis. □

Corollary 2.2.11
The Lie algebra $\mathbb{E}\otimes L$ is nilpotent if and only if the Lie algebra L is nilpotent.

Proposition 2.2.29
We have $D(\mathbb{E}\otimes L)=\mathbb{E}\otimes DL$ for $\forall n$.

Proof. We have

$$\begin{aligned} D(\mathbb{E}\otimes L) &= \operatorname{span}[\mathbb{E}\otimes L, \mathbb{E}\otimes L] \\ &= \operatorname{span}[L, L] \\ &= \operatorname{span} DL \\ &= \mathbb{E}\otimes DL. \end{aligned}$$ □

Corollary 2.2.12
The Lie algebra $\mathbb{E}\otimes L$ is solvable if and only if the Lie algebra L is solvable.

Theorem 2.2.6 (Engel)
A finite dimensional Lie algebra over a field of characteristic 0 is solvable if and only if the derived Lie algebra is nilpotent.

Proof. We may assume that the underlying field $\mathbb{F}$ is algebraically closed. Suppose that L is solvable. We may assume that $\operatorname{ad} L$ is a Lie subalgebra of the subalgebra

$$\bigl\{\, x\in M(n,\mathbb{F}) : x_{ij}=0 \text{ for } \forall i>\forall j \,\bigr\}$$

by Lie's theorem. The Lie subalgebra $\operatorname{ad} DL$ is contained in the subalgebra

$$\bigl\{\, x\in M(n,\mathbb{F}) : x_{ij}=0 \text{ for } \forall i\geq\forall j \,\bigr\}.$$

The Lie algebra DL is nilpotent by Corollary 2.1.3. □

2.3 Jordan Decomposition of a Linear Mapping

Suppose that V is a finite dimensional vector space over a field $\mathbb{F}$ and let $\bar{x}$ be an element of $\hom V$. We write $\mathbb{F}[x]$ for the algebra of polynomials in one indeterminate x over $\mathbb{F}$.

Proposition 2.3.1
The mapping $f(x) \mapsto f(\bar{x})$ is a homomorphism of algebras of $\mathbb{F}[x]$ into $\hom V$.

Definition 2.3.1
The unique monic polynomial generating the ideal

$$\left\{ f(x) \in \mathbb{F}[x] : f(\bar{x}) = 0 \right\} \neq \{0\}$$

is called the minimal polynomial of $\bar{x}$.

We write $\overline{\mathbb{F}}$ for the algebraic closure of $\mathbb{F}$ and the minimal polynomial of $\bar{x}$ by

$$f_0(x) = \prod_{\nu \in \overline{\mathbb{F}}} (x - \nu)^{m(\nu)}.$$

Proposition 2.3.2
We have

$$\frac{\mathbb{F}[x]}{\mathbb{F}[x] f_0(x)} = \left\{ f(\bar{x}) : f(x) \in \mathbb{F}[x] \right\}.$$

Proposition 2.3.3
An element $\bar{x}$ of $\hom V$ is diagonalisable if and only if

$$V = \bigoplus_{\nu \in \mathbb{F}} \ker(\bar{x} - \nu).$$

Proof. Suppose that $\bar{x}$ is diagonalisable and let

$$\{\nu_k\}_{k=1}^n = \left\{ \nu \in \mathbb{F} : \ker(\bar{x} - \nu) \neq \{0\} \right\}$$

such that $\#\{\nu_k\}_{k=1}^n = n$. We have

$$\begin{aligned} V &= \bigoplus_{k=1}^n \ker(\bar{x} - \nu_k) \\ &= \bigoplus_{\nu \in \mathbb{F}} \ker(\bar{x} - \nu). \end{aligned}$$

Suppose that

$$V = \bigoplus_{\nu \in \mathbb{F}} \ker(\bar{x} - \nu)$$

and let

$$\{\nu_k\}_{k=1}^n = \left\{ \nu \in \mathbb{F} : \ker(\bar{x} - \nu) \neq \{0\} \right\}$$

such that $\#\{\nu_k\}_{k=1}^n = n$. We have

$$V = \bigoplus_{k=1}^n \ker(\bar{x} - \nu_k). \quad \square$$

Definition 2.3.2

$$\bar{x} = S + N$$

is called a Jordan decomposition if it satisfies the following.

1. An element S of $\hom V$ is diagonalisable.

2. An element N of $\hom V$ is nilpotent.

3. We have $[S, N] = 0$.

Proposition 2.3.4
Suppose that N is a nilpotent element of $\hom V$. We have

$$(1-N)^{-1} = \sum_{n=0}^{\infty} N^n.$$

Proof. We define $m = \min\{\, m : N^m = 0 \,\}$. We have

$$\begin{aligned}(1-N)\sum_{n=0}^{\infty} N^n &= (1-N)\sum_{n<m} N^n \\ &= 1 - N^m = 1.\end{aligned}$$ □

Theorem 2.3.1
Suppose that $\bar{x} = S + N$ is a Jordan decomposition. We have the following.

1. The set $\left\{\, \nu \in \overline{\mathbb{F}} : f_0(\nu) = 0 \,\right\}$ is contained in $\mathbb{F}$.

2. We have $m(\nu) = \min\left\{\, m : N^m \ker(S-\nu) = \{0\} \,\right\}$ for $\forall \nu$ of $\mathbb{F}$.

3. We have $\ker(S-\nu) = \ker(\bar{x}-\nu)^{m(\nu)}$ for $\forall \nu$ of $\mathbb{F}$.

4. We have
$$V = \bigoplus_{\nu \in \mathbb{F}} \ker(\bar{x}-\nu)^{m(\nu)}$$
and the element $\bar{x}$ has a unique Jordan decomposition $\bar{x} = S + N$.

Proof. We have
$$V = \bigoplus_{\nu \in \mathbb{F}} \ker(S-\nu)$$
by Proposition 2.3.3 and let

$$\{\nu_k\}_{k=1}^n = \left\{\, \nu \in \mathbb{F} : \ker(S-\nu) \neq \{0\} \,\right\}$$

such that $\#\{\nu_k\}_{k=1}^n = n$. The subspace $\ker(S-\nu_k)$ is invariant for $\bar{x}$ for $\forall k$ since $\bar{x}$ and S commute. We define

$$\begin{aligned} m_k &= \min\left\{\, m : N^m \ker(S-\nu_k) = \{0\} \,\right\} \\ &= \min\left\{\, m : (\bar{x}-\nu_k)^m \ker(S-\nu_k) = \{0\} \,\right\} \geq 1 \end{aligned}$$

for $\forall k$. The minimal polynomial of the restriction of $\bar{x}$ to the invariant subspace $\ker(S-\nu_k)$ is $(x-\nu_k)^{m_k}$ and thus $(x-\nu_k)^{m_k}$ divides $f_0(x)$ for $\forall k$. We have

$$f_0(x) = \prod_{k=1}^{n}(x-\nu_k)^{m_k}$$

since

$$\prod_{k=1}^{n}(\bar{x}-\nu_k)^{m_k} = 0.$$

The subspace $\ker(S-\nu_k)$ is contained in $\ker(\bar{x}-\nu_k)^{m_k}$ for $\forall k$. Suppose that $1 \le k_0 \le n$ and let

$$v = \sum_{k=1}^{n} v_k \in \bigoplus_{k=1}^{n} \ker(S-\nu_k) = V$$

be an element of $\ker(\bar{x}-\nu_{k_0})^{m_{k_0}}$. We have

$$\begin{aligned} 0 &= (\bar{x}-\nu_{k_0})^{m_{k_0}} v \\ &= \sum_{k=1}^{n}(\bar{x}-\nu_{k_0})^{m_{k_0}} v_k \end{aligned}$$

and

$$\begin{aligned} 0 &= (\bar{x}-\nu_{k_0})^{m_{k_0}} v_k \\ &= (\nu_k-\nu_{k_0}+N)^{m_{k_0}} v_k \end{aligned}$$

for $\forall k$ since the subspace $\ker(S-\nu_k)$ is invariant for $(\bar{x}-\nu_{k_0})^{m_{k_0}}$ for $\forall k$. We have $v_k = 0$ provided that $k \neq k_0$ since $\nu_k - \nu_{k_0} + N$ is invertible by Proposition 2.3.4. We have $\ker(S-\nu_k) = \ker(\bar{x}-\nu_k)^{m_k}$ for $\forall k$. Suppose that

$$v = \sum_{k=1}^{n} v_k \in \bigoplus_{k=1}^{n} \ker(\bar{x}-\nu_k)^{m_k} = V.$$

We have $Sv = \sum_{k=1}^{n} \nu_k v_k$. □

Definition 2.3.3

Suppose that S is a subset of a commutative ring R. We have the following.

1. An element d of R is called a common divisor of S if S is a subset of Rd.

2. A common divisor d of S is called a greatest common divisor if d is a multiple of any common divisor of S.

Proposition 2.3.5

Any subset of a principal ideal domain has a greatest common divisor.

Proof. Suppose that S is a subset of a principal ideal domain R. There exists an element d of R such that $(S) = Rd$. The element d is a common divisor of S contained in (S). □

Proposition 2.3.6
Suppose that S is a subset of a principal ideal domain R.

$$\{\text{a greatest common divisor of } S\} = \{\, d \in R : Rd = (S) \,\}.$$

Proposition 2.3.7
Suppose that R is a commutative ring with identity and let S be a subset of R. Suppose that d_0 is a greatest common divisor of S.

$$\{\text{a greatest common divisor of } S\} = \{\, d \in R : Rd = Rd_0 \,\}.$$

Proposition 2.3.8
Suppose that S is a subset of an integral domain R and let d_0 be a greatest common divisor of S.

$$\{\text{a greatest common divisor of } S\} = R^{\times} d_0.$$

Definition 2.3.4
A subset S of an integral domain R is said to be relatively prime if

$$\{\text{a greatest common divisor of } S\} = R^{\times}.$$

Proposition 2.3.9
A subset S of a principal ideal domain R is relatively prime if and only if

$$R = (S).$$

Corollary 2.3.1
A subset $\{x_1, \ldots, x_n\}$ of a principal ideal domain R is relatively prime if and only if

$$R = Rx_1 + \cdots + Rx_n.$$

Proposition 2.3.10
Suppose that S is a subset of $\mathbb{F}[x]$ such that $S \setminus \{0\}$ is not empty. There exists a unique monic greatest common divisor of S.

Proof. There exists a greatest common divisor $d(x)$ of S by Proposition 2.3.5. We have $d(x) \neq 0$ since $S \setminus \{0\}$ is not empty. We have

$$\{\text{a greatest common divisor of } S\} = \big(\mathbb{F} \setminus \{0\}\big) d(x)$$

by Proposition 2.3.8. □

Suppose that the set
$$\{\, \nu \in \overline{\mathbb{F}} : f_0(\nu) = 0 \,\}$$
is contained in $\mathbb{F}$ and let $\{\nu_k\}_{k=1}^n$ be a subset of $\mathbb{F}$ containing the set
$$\{\, \nu \in \overline{\mathbb{F}} : f_0(\nu) = 0 \,\}$$
such that $\#\{\nu_k\}_{k=1}^n = n$.

There exists a set $\{m_k\}_{k=1}^n$ of positive integers such that $f_0(x)$ divides
$$f(x) = \prod_{k=1}^n (x - \nu_k)^{m_k}.$$

Proposition 2.3.11

There exists an element $\big(f_1(x), \dots, f_n(x)\big)_{k=1}^n$ of
$$\mathbb{F}[x]\frac{f(x)}{(x-\nu_1)^{m_1}} \times \cdots \times \mathbb{F}[x]\frac{f(x)}{(x-\nu_n)^{m_n}}$$
such that $1 = \sum_{k=1}^n f_k(\bar{x})$.

Proof. We may assume that $n \geq 1$. The set
$$\left\{ \frac{f(x)}{(x-\nu_1)^{m_1}}, \dots, \frac{f(x)}{(x-\nu_n)^{m_n}} \right\}$$
is relatively prime. There exists an element $\big(f_1(x), \dots, f_n(x)\big)_{k=1}^n$ of
$$\mathbb{F}[x]\frac{f(x)}{(x-\nu_1)^{m_1}} \times \cdots \times \mathbb{F}[x]\frac{f(x)}{(x-\nu_n)^{m_n}}$$
such that $1 = \sum_{k=1}^n f_k(x)$ by Corollary 2.3.1. $\square$

Proposition 2.3.12

Suppose that M is a left module over a ring and let Λ be a set. Suppose that $(P_i)_{i\in\Lambda}$ is an element of $(\hom M)^\Lambda$ satisfying the following.

1. The element $\big(P_i(x)\big)_{i\in\Lambda}$ belongs to $\bigoplus_{i\in\Lambda} M$ and we have
$$x = \sum_{i\in\Lambda} P_i(x)$$
for $\forall x$.

2. We have $P_i \circ P_j = 0$ provided that $i \neq j$.

We have the following.

1. We have $P_i \circ P_j = \delta_{ij} P_i$ and
$$P_i(M) = \{\, x \in M : P_i(x) = x \,\}$$
for $\forall i$ and $\forall j$.

2. We have
$$M = \bigoplus_{i\in\Lambda} P_i(M).$$

Proof. 1. Suppose that x is an element of M. We have $P_i(x) = P_i^2(x)$ since
$$x = P_i(x) + \sum_{j\in\Lambda\setminus\{i\}} P_j(x).$$

2. Suppose that x is an element of M. The element $\bigoplus_{i\in\Lambda} P_i(x)$ belongs to $\bigoplus_{i\in\Lambda} P_i(M)$ and $x = \sum_{i\in\Lambda} P_i(x)$. Suppose that $\bigoplus_{i\in\Lambda} x_i$ is an element of $\bigoplus_{i\in\Lambda} P_i(M)$ such that $x = \sum_{i\in\Lambda} x_i = 0$. We have
$$\begin{aligned} 0 &= P_i(x) \\ &= P_i(x_i) \\ &= x_i \end{aligned}$$
for $\forall i$. $\square$

PROPOSITION 2.3.13
We have
$$V = \bigoplus_{k=1}^{n} \ker(\bar{x} - \nu_k)^{m_k}$$
and $f_i(\bar{x})f_j(\bar{x}) = \delta_{ij} f_i(\bar{x})$.

Proof. Suppose that $i \neq j$. The polynomial $f_i(x)f_j(x)$ belongs to $\mathbb{F}[x]f(x)$ since $\big(f_i(x), f_j(x)\big)$ belongs to
$$\mathbb{F}[x]\frac{f(x)}{(x-\nu_i)^{m_i}} \times \mathbb{F}[x](x-\nu_i)^{m_i}.$$
We have $f_i(\bar{x})f_j(\bar{x}) = 0$. We have
$$V = \bigoplus_{k=1}^{n} f_k(\bar{x})V$$
by Proposition 2.3.12. The subspace $f_k(\bar{x})V$ is contained in $\ker(\bar{x} - \nu_k)^{m_k}$ for $\forall k$ since $(\bar{x} - \nu_k)^{m_k} f_k(\bar{x}) = 0$. Suppose that $1 \leq k_0 \leq n$ and let v be an element of $\ker(\bar{x} - \nu_{k_0})^{m_{k_0}}$. We have
$$\begin{aligned} v &= f_{k_0}(\bar{x})v + \sum_{k\neq k_0} f_k(\bar{x})v \\ &= f_{k_0}(\bar{x})v \end{aligned}$$
since $f_k(x)$ belongs to $\mathbb{F}[x](x-\nu_{k_0})^{m_{k_0}}$ provided that $k \neq k_0$. $\square$

Proposition 2.3.14
We have the following.

1. We have

$$\begin{aligned} V &= \bigoplus_{\nu \in \mathbb{F}} \ker(\bar{x} - \nu)^{m(\nu)} \\ &= \bigoplus_{\nu \in \mathbb{F}} \lim_{m \to \infty} \ker(\bar{x} - \nu)^{m}. \end{aligned}$$

2. We write P_ν for the projection of V onto $\lim_{m\to\infty} \ker(\bar{x} - \nu)^m$ for $\forall \nu$. We have $f_k(\bar{x}) = P_{\nu_k}$ for $\forall k$.

3. Suppose that ν_0 is an element of $\mathbb{F}$ and let ν be an element of $\mathbb{F} \setminus \{\nu_0\}$. The projection P_ν belongs to

$$\lim_{n \to \infty} \left\{ f(\bar{x}) : f(x) \in \mathbb{F}[x](x - \nu_0)^n \right\}.$$

Theorem 2.3.2
The set $\left\{ \nu \in \overline{\mathbb{F}} : f_0(\nu) = 0 \right\}$ is contained in $\mathbb{F}$ if and only if $\bar{x}$ is Jordan decomposable.

Proof. We define

$$\begin{aligned} S &= \sum_{k=1}^{n} \nu_k f_k(\bar{x}) \\ &= \sum_{k=1}^{n} \nu_k P_{\nu_k} \\ &= \sum_{\nu \in \mathbb{F}} \nu P_\nu \end{aligned}$$

and

$$\begin{aligned} N &= \bar{x} - S \\ &= \sum_{k=1}^{n} (\bar{x} - \nu_k) f_k(\bar{x}). \end{aligned}$$

The element S is diagonalisable and we have $[S, N] = 0$. We write $m = \sup\{m_k\}_{k=1}^{n}$. We have

$$\begin{aligned} N^m &= \sum_{k=1}^{n} (\bar{x} - \nu_k)^m f_k(\bar{x}) \\ &= 0. \end{aligned}$$

□

Corollary 2.3.2
We have the following.

1. The element S belongs to

$$\lim_{n\to\infty}\left\{\, f(\bar{x}) : f(x) \in \mathbb{F}[x]x^n \,\right\}.$$

2. The element N belongs to $\left\{\, f(\bar{x}) : f(x) \in \mathbb{F}[x]x \,\right\}$.

Corollary 2.3.3
An element $\bar{x}$ of $\hom V$ is diagonalisable if and only if $\bar{x}$ is Jordan decomposable and the minimal polynomial of $\bar{x}$ does not have a multiple root.

Proof. We write 1_m for the identity of $M(m,\mathbb{F})$ for $\forall m$. Suppose that

$$\bar{x} = \operatorname{diag}(\nu_1 1_{m_1}, \ldots, \nu_n 1_{m_n}),$$

where $\{\nu_k\}_{k=1}^n$ is a subset of $\mathbb{F}$ such that $\#\{\nu_k\}_{k=1}^n = n$ and $\{m_k\}_{k=1}^n$ is a subset of $\mathbb{N}$. The minimal polynomial of $\bar{x}$ is

$$(x-\nu_1)\cdots(x-\nu_n).$$

Suppose that the minimal polynomial of $\bar{x}$ is

$$(x-\nu_1)\cdots(x-\nu_n),$$

where $\{\nu_k\}_{k=1}^n$ is a subset of $\mathbb{F}$ such that $\#\{\nu_k\}_{k=1}^n = n$. We have

$$V = \bigoplus_{k=1}^{n} \ker(\bar{x}-\nu_k)$$

by Proposition 2.3.14 □

Corollary 2.3.4
A restriction of a diagonalisable element of $\hom V$ to an invariant subspace is diagonalisable.

Proof. Suppose that $\bar{x}$ is a diagonalisable element of $\hom V$ and let V_0 be an invariant subspace. The minimal polynomial of the restriction of the element $\bar{x}$ to the invariant subspace V_0 divides the minimal polynomial of the element $\bar{x}$. □

Theorem 2.3.3
A subset S of the set

$$\left\{\text{a diagonalisable element of } \hom V\right\}$$

is simultaneously diagonalisable if and only if $[\bar{x}_1, \bar{x}_2] = 0$ for $\forall(\bar{x}_1, \bar{x}_2)$ of S^2.

Proof. Suppose that $[\bar{x}_1, \bar{x}_2] = 0$ for $\forall(\bar{x}_1, \bar{x}_2)$ of S^2. The proof is by induction on $n = \dim \operatorname{span} S$. Suppose that $n > 0$ and let $\{\bar{x}_1, \dots, \bar{x}_n\}$ be a basis of $\operatorname{span} S$ contained in S. We have

$$V = \bigoplus_{(\nu_1,\dots,\nu_{n-1}) \in \mathbb{F}^{n-1}} \ker(\bar{x}_1 - \nu_1) \cap \cdots \cap \ker(\bar{x}_{n-1} - \nu_{n-1})$$

since $\{\bar{x}_1, \dots, \bar{x}_{n-1}\}$ is simultaneously diagonalisable by the induction hypothesis. Suppose that $\nu = (\nu_1, \dots, \nu_{n-1})$ is an element of $\mathbb{F}^{n-1}$. The subspace

$$V_\nu = \ker(\bar{x}_1 - \nu_1) \cap \cdots \cap \ker(\bar{x}_{n-1} - \nu_{n-1})$$

is invariant for $\bar{x}_n$ since $[\bar{x}_1, \bar{x}_n] = \cdots = [\bar{x}_{n-1}, \bar{x}_n] = 0$. The restriction of $\bar{x}_n$ to V_ν is diagonalisable by Corollary 2.3.4. The set $\{\bar{x}_1, \dots, \bar{x}_n\}$ is simultaneously diagonalisable. The set $\operatorname{span} S$ is simultaneously diagonalisable. □

Theorem 2.3.4

A mapping f of a finite subset S of $\mathbb{F}$ into $\mathbb{F}$ extends uniquely to an element $f(x)$ of $\mathbb{F}[x]$ such that $\deg f(x) < \#S = n$. We have

$$f(x) = f(\nu_1)\frac{x-\nu_2}{\nu_1-\nu_2}\cdots\frac{x-\nu_n}{\nu_1-\nu_n} + \cdots + f(\nu_n)\frac{x-\nu_1}{\nu_n-\nu_1}\cdots\frac{x-\nu_{n-1}}{\nu_n-\nu_{n-1}},$$

where $S = \{\nu_k\}_{k=1}^n$.

Proof. The proof of uniqueness is by induction on n. Suppose that $n > 0$. Suppose that f extends to an element

$$f(x) = f_1(x)(x - \nu_1) + f(\nu_1)$$

such that $\deg f_1(x) < n - 1$. We have

$$\begin{aligned} f_1(x) = {} & \frac{f(\nu_2) - f(\nu_1)}{\nu_2 - \nu_1}\frac{x-\nu_3}{\nu_2-\nu_3}\cdots\frac{x-\nu_n}{\nu_2-\nu_n} + \cdots \\ & + \frac{f(\nu_n) - f(\nu_1)}{\nu_n - \nu_1}\frac{x-\nu_2}{\nu_n-\nu_2}\cdots\frac{x-\nu_{n-1}}{\nu_n-\nu_{n-1}} \end{aligned}$$

by the induction hypothesis. □

Corollary 2.3.5

Suppose that $\bar{x}$ is a diagonalisable element of $\hom V$. We have

$$\prod_{\det(\bar{x}-\nu)=0} \mathbb{F} = \left\{\, f(\bar{x}) : f(x) \in \mathbb{F}[x] \,\right\}.$$

2.4 Cartan's Criteria

Suppose that V is a finite dimensional vector space over a field $\mathbb{F}$.

Proposition 2.4.1
$\operatorname{ad} x$ is diagonalisable if x is a diagonalisable element of $\hom V$.

Proof. There exists a basis $\{e_k\}_{k=1}^n$ such that

$$x(e_1, \ldots, e_n) = (e_1, \ldots, e_n) \operatorname{diag}(\nu_1, \ldots, \nu_n).$$

We have

$$(\operatorname{ad} x)(e_{ij}) = [\sum_{k=1}^{n} \nu_k e_{kk}, e_{ij}] = (\nu_i - \nu_j) e_{ij}. \quad \square$$

Corollary 2.4.1
Suppose that $x = S + N$ is a Jordan decomposition of an element of $\hom V$.

$$\operatorname{ad} x = \operatorname{ad} S + \operatorname{ad} N$$

is a Jordan decomposition.

Proof. The element $\operatorname{ad} N$ is nilpotent by Proposition 2.1.8 and we have

$$[\operatorname{ad} S, \operatorname{ad} N] = \operatorname{ad}[S, N] = 0. \quad \square$$

Suppose that ρ is a representation of a Lie algebra on a vector space.

Definition 2.4.1
A bilinear form f is said to be invariant for ρ if

$$f\big(\rho(x)v, w\big) + f\big(v, \rho(x)w\big) = 0$$

for $\forall(x, v, w)$.

Proposition 2.4.2
Suppose that ρ is finite dimensional. A symmetric form

$$B_\rho(x, y) = \operatorname{tr}\big(\rho(x)\rho(y)\big)$$

is invariant for the adjoint representation.

Proof. We have

$$\begin{aligned} B_\rho\big((\operatorname{ad} z)(x), y\big) &= \operatorname{tr}\Big(\big[\rho(z), \rho(x)\big]\rho(y)\Big) \\ &= -\operatorname{tr}\Big(\rho(x)\big[\rho(z), \rho(y)\big]\Big) \\ &= -B_\rho\big(x, (\operatorname{ad} z)(y)\big). \end{aligned} \quad \square$$

Corollary 2.4.2
The Killing form on a finite dimensional Lie algebra is invariant for the adjoint representation.

Proposition 2.4.3
Suppose that B is an invariant bilinear form on a Lie algebra L and let L_0 be an ideal of L. The subspace $\{\, x \in L : B(x, L_0) = \{0\} \,\}$ is an ideal of L.

Proof. Suppose that $B(x_0^{\perp}, L_0) = \{0\}$. We have

$$B\Big([x, x_0^{\perp}], x_0\Big) = -B\Big(x_0^{\perp}, [x, x_0]\Big) = 0$$

for $\forall (x, x_0)$ of $L \times L_0$. $\square$

Proposition 2.4.4
Suppose that $\mathbb{F}$ is an algebraically closed field of characteristic 0. We define

$$\begin{aligned}\mathcal{L} &= \{\, \bar{x} \in \hom V : (\operatorname{ad} \bar{x})A \text{ is a subset of } B \,\} \\ &= \{\, \bar{x} \in \hom V : f(\operatorname{ad} \bar{x})A \text{ is a subset of } B \text{ for } \forall f(x) \text{ of } \mathbb{F}[x]x \,\},\end{aligned}$$

where A is a subspace of $\hom V$ and B is a subspace of A. An element of

$$\mathcal{L}^{\perp} = \{\, \bar{x} \in \mathcal{L} : \operatorname{tr}(\bar{x}\mathcal{L}) = \{0\} \,\}$$

is nilpotent and the Lie algebra $\mathcal{L}^{\perp}$ is nilpotent.

Proof. Suppose that $\bar{x}$ is an element of $\mathcal{L}^{\perp}$ and let $\bar{x} = S + N$ be the Jordan decomposition. The element S belongs to $\mathcal{L}$ since there exists an element $f(x)$ of $\mathbb{F}[x]x$ such that $\operatorname{ad} S = f(\operatorname{ad} \bar{x})$. There exists a basis $\{e_k\}_{k=1}^{n}$ such that

$$S(e_1, \ldots, e_n) = (e_1, \ldots, e_n) \operatorname{diag}(\nu_1, \ldots, \nu_n).$$

Suppose that f is an arbitrary linear functional on $\langle \nu_1, \ldots, \nu_n \rangle_{\mathbb{Q}}$. The element $f(S)$ belongs to $\mathcal{L}$ since

$$\begin{aligned}\big(\operatorname{ad} f(S)\big)(e_{ij}) &= \big(f(\nu_i) - f(\nu_j)\big)e_{ij} \\ &= f(\nu_i - \nu_j)e_{ij} \\ &= f(\operatorname{ad} S)(e_{ij}).\end{aligned}$$

We have

$$\begin{aligned}\sum_{k=1}^{n} f(\nu_k)^2 &= f\Big(\operatorname{tr}\big(Sf(S)\big)\Big) \\ &= f\Big(\operatorname{tr}\big(\bar{x}f(S)\big) - \operatorname{tr}\big(Nf(S)\big)\Big) = 0\end{aligned}$$

since $Nf(S)$ is nilpotent. $\square$

Theorem 2.4.1 (Chevalley)
Suppose that $\mathbb{F}$ is a field of characteristic 0. A Lie subalgebra L of $\hom V$ is solvable if

$$\operatorname{tr}(L \cdot DL) = \{0\}.$$

Proof. We may assume that $\mathbb{F}$ is algebraically closed. It is sufficient to show that the derived Lie algebra is nilpotent. The Lie algebra L is a Lie subalgebra of the Lie algebra

$$\mathcal{L} = \left\{ \bar{x} \in \hom V : (\operatorname{ad} \bar{x})L \text{ is a subset of } DL \right\}.$$

We have

$$\operatorname{tr}(\mathcal{L} \cdot DL) = \{0\}.$$

The derived Lie algebra is a Lie subalgebra of the nilpotent Lie algebra $\mathcal{L}^{\perp}$. □

Theorem 2.4.2
A finite dimensional Lie algebra L over a field of characteristic 0 is solvable if and only if

$$\operatorname{tr}\big((\operatorname{ad} L)(\operatorname{ad} DL)\big) = \{0\}.$$

Proof. We may assume that the underlying field is algebraically closed.

Suppose that L is solvable. We may assume that $\operatorname{ad} L$ is a Lie subalgebra of the subalgebra

$$\left\{ x \in M(n, \mathbb{F}) : x_{ij} = 0 \text{ for } \forall i > \forall j \right\}$$

by Lie's theorem. The Lie subalgebra $\operatorname{ad} DL$ is contained in the subalgebra

$$\left\{ x \in M(n, \mathbb{F}) : x_{ij} = 0 \text{ for } \forall i \geq \forall j \right\}.$$

Suppose that

$$\operatorname{tr}\big((\operatorname{ad} L)(\operatorname{ad} DL)\big) = \{0\}.$$

The Lie algebra L is solvable since the Lie subalgebra $\operatorname{ad} L$ is solvable by Chevalley's theorem. □

Corollary 2.4.3 (Cartan)
A finite dimensional Lie algebra over a field of characteristic 0 is solvable if the Killing form is trivial.

Proposition 2.4.5
A finite dimensional Lie algebra has a maximal solvable ideal.

Proof. There exists a solvable ideal L_0 such that

$$\dim L_0 = \max\left\{ \dim L_0 : L_0 \text{ is a solvable ideal} \right\}$$

since the ideal $\{0\}$ is solvable. □

Theorem 2.4.3
Suppose that L_0 is an ideal of a Lie algebra L and let L_1 be a Lie subalgebra of L. We have

$$\frac{L_1}{L_0 \cap L_1} = \frac{L_0 + L_1}{L_0}.$$

Proof. The subspace $L_0 + L_1$ is a Lie subalgebra. □

Proposition 2.4.6
Suppose that L_0 is an ideal of a Lie algebra L. The Lie algebra L is solvable if and only if the ideal L_0 and the quotient Lie algebra L/L_0 are solvable.

Proof. Suppose that $D^m L_0 = \{0\}$ and $D^n(L/L_0) = \{0\}$. We have $D^{m+n}L = \{0\}$ since the ideal $D^n L$ is contained in the ideal L_0. □

Theorem 2.4.4
The set of solvable ideals of a Lie algebra is a directed set.

Proof. The set of solvable ideals is not empty since the ideal $\{0\}$ is solvable. Suppose that L_1 and L_2 are solvable ideals. The ideal $L_1 + L_2$ is solvable since the ideal L_1 and the quotient Lie algebra

$$\frac{L_1 + L_2}{L_1} = \frac{L_2}{L_1 \cap L_2}$$

are solvable. □

Proposition 2.4.7
A maximal element of a directed set is the maximum element.

Corollary 2.4.4
A maximal solvable ideal of a Lie algebra is the maximum solvable ideal.

Definition 2.4.2
The maximum solvable ideal of a Lie algebra is called the radical of the Lie algebra. The radical of a Lie algebra L is denoted by $\operatorname{rad} L$.

Corollary 2.4.5
A finite dimensional Lie algebra has the radical.

Definition 2.4.3
A Lie algebra is said to be semisimple if the ideal $\{0\}$ is the only solvable ideal.

Theorem 2.4.5
A Lie algebra is semisimple if and only if the ideal $\{0\}$ is the only commutative ideal.

Proof. Suppose that the ideal $\{0\}$ is the only commutative ideal and let L be an arbitrary solvable ideal. Suppose that $n = \min\{\, n : D^n L = \{0\}\,\} > 0$. The ideal $D^{n-1}L$ is commutative. This is a contradiction. □

Proposition 2.4.8
Suppose that L_0 is an ideal of a finite dimensional Lie algebra L. We have

$$\operatorname{ad} x = \begin{pmatrix} \operatorname{ad}_{L_0} x & * \\ 0 & \operatorname{ad}(x + L_0) \end{pmatrix}$$

for $\forall x$.

PROPOSITION 2.4.9
Any commutative ideal of a finite dimensional Lie algebra L is contained in the ideal $L^{\perp}$.

Proof. Suppose that L_0 is a commutative ideal of the Lie algebra L and let x_0 be an element of the ideal L_0. We have

$$\operatorname{ad} x_0 = \begin{pmatrix} \operatorname{ad}_{L_0} x_0 & * \\ 0 & \operatorname{ad}(x_0 + L_0) \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}. \quad \square$$

THEOREM 2.4.6
A finite dimensional Lie algebra is semisimple if the Killing form is nondegenerate.

Proof. We have $L^{\perp} = \{0\}$ since the Killing form is nondegenerate. □

THEOREM 2.4.7 (Cartan)
The following are equivalent for a finite dimensional Lie algebra over a field of characteristic 0.

1. The Lie algebra is semisimple.
2. The Killing form is nondegenerate.
3. We have
$$L_0 \cap L_0^{\perp} = \{\, x \in L_0 : B(x, L_0) = \{0\} \,\} = \{0\}$$
for an arbitrary ideal L_0.

Proof. Suppose that the Lie algebra is semisimple and let L_0 be an arbitrary ideal. The Killing form on the ideal $L_0 \cap L_0^{\perp}$ is trivial. The ideal $L_0 \cap L_0^{\perp}$ is solvable by Cartan's criterion for solvability. □

COROLLARY 2.4.6
Any ideal of a finite dimensional semisimple Lie algebra over a field of characteristic 0 is semisimple.

Proof. Suppose that L_0 is an arbitrary ideal of a finite dimensional semisimple Lie algebra over a field of characteristic 0. The Killing form on the ideal L_0 is nondegenerate since we have $L_0 \cap L_0^{\perp} = \{0\}$. The ideal L_0 is semisimple. □

2.5 Cohomology

THEOREM 2.5.1
We have

$$\hom(\bigwedge^n M, N) = \{\text{an alternating mapping of } M^n \text{ into } N\}$$

for unital modules M and N over a commutative ring with identity.

Proposition 2.5.1
We have
$$\prod_i \hom(M_i, N) = \hom(\bigoplus_i M_i, N)$$
for unital modules M_i and N over a commutative ring with identity.

Suppose that ρ is a representation of a Lie algebra L on a vector space V and let f be an element of $\hom(\bigwedge^n L, V)$. We define

$$\begin{aligned}\partial f(x_1, \ldots, x_{n+1}) = &-\sum_{k=1}^{n+1} (-1)^k \rho(x_k) f(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}) \\ &+ \sum_{i<j} (-1)^{i+j} f\Big(\big[x_i, x_j\big], x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}\Big).\end{aligned}$$

We write
$$x^k = x_1 \wedge \cdots \wedge x_{k-1} \wedge x_{k+1} \wedge \cdots \wedge x_n$$
for $\forall k$ and
$$x^{ij} = \big[x_i, x_j\big] \wedge x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_n$$
for $\forall i < \forall j$.

Proposition 2.5.2
We have
$$\hom(\bigwedge L, V) = \prod_{n=0}^{\infty} \hom(\bigwedge^n L, V).$$

Proposition 2.5.3
The element ∂ belongs to $\hom\hom(\bigwedge L, V)$.

$$\hom(\bigwedge^0 L, V) \xrightarrow{\ \partial\ } \hom(\bigwedge^1 L, V) \xrightarrow{\ \partial\ } \hom(\bigwedge^2 L, V) \xrightarrow{\ \partial\ } \cdots$$

Proof. Suppose that $i < j$ and let $x_i = x_j = x$. We have
$$\partial f(x_1, \ldots, x_{n+1}) = C_1 + C_2,$$
where
$$C_1 = -\rho(x)\Big((-1)^i f\big(x^i\big) + (-1)^j f\big(x^j\big)\Big) = 0$$
and
$$\begin{aligned}C_2 = &\sum_{k<i} \Big((-1)^{k+i} f\big(x^{ki}\big) + (-1)^{k+j} f\big(x^{kj}\big)\Big) \\ &+ \sum_{i<k<j} \Big((-1)^{i+k} f\big(x^{ik}\big) + (-1)^{k+j} f\big(x^{kj}\big)\Big) \\ &+ \sum_{j<k} \Big((-1)^{i+k} f\big(x^{ik}\big) + (-1)^{j+k} f\big(x^{jk}\big)\Big) = 0. \quad \square\end{aligned}$$

We define

$$\theta(x)f(x_1,\ldots,x_n)=\rho(x)f(x_1,\ldots,x_n) - \sum_{k=1}^{n} f\Big(x_1,\ldots,x_{k-1},\big[x,x_k\big],x_{k+1},\ldots,x_n\Big).$$

Proposition 2.5.4
The element $\theta(x)$ belongs to hom hom$(\bigwedge L, V)$ for $\forall x$.

Proof. Suppose that $i<j$ and let $x_i=x_j$. We have

$$\theta(x)f(x_1,\ldots,x_n)=-f\Big(x_1,\ldots,x_{i-1},\big[x,x_i\big],x_{i+1},\ldots,x_n\Big) - f\Big(x_1,\ldots,x_{j-1},\big[x,x_j\big],x_{j+1},\ldots,x_n\Big)=0. \quad \square$$

Suppose that f is an element of $\hom(\bigwedge^{n+1} L, V)$. We define

$$\iota(x)f(x_1,\ldots,x_n)=f(x,x_1,\ldots,x_n).$$

Proposition 2.5.5
The element $\iota(x)$ belongs to hom hom$(\bigwedge L, V)$.

$$0 \xleftarrow{\iota(x)} \hom(\textstyle\bigwedge^0 L,V) \xleftarrow{\iota(x)} \hom(\textstyle\bigwedge^1 L,V) \xleftarrow{\iota(x)} \cdots$$

Proposition 2.5.6
We have $\theta(x)=\big\{\iota(x),\partial\big\}$ for $\forall x$.

Proof. We have

$$\begin{aligned}\iota(x)\partial f(x_1,\ldots,x_n)&=\partial f(x,x_1,\ldots,x_n)\\ &=\theta(x)f(x_1,\ldots,x_n)+\sum_{k=1}^{n}(-1)^k\rho(x_k)f\big(x\wedge x^k\big)-\sum_{i<j}(-1)^{i+j}f\big(x\wedge x^{ij}\big)\end{aligned}$$

and

$$\partial\iota(x)f(x_1,\ldots,x_n)=-\sum_{k=1}^{n}(-1)^k\rho(x_k)f\big(x\wedge x^k\big)+\sum_{i<j}(-1)^{i+j}f\big(x\wedge x^{ij}\big).$$

We have $\theta(x)f(x_1,\ldots,x_n)=\big\{\iota(x),\partial\big\}f(x_1,\ldots,x_n)$. $\square$

Proposition 2.5.7
We have $\iota\big([x_1,x_2]\big)=\big[\theta(x_1),\iota(x_2)\big]$ for $\forall(x_1,x_2)$.

Proof. We have

$$\theta(x_1)\iota(x_2)f(x_3,\ldots,x_{n+2})=\rho(x_1)f(x_2,\ldots,x_{n+2})-\sum_{k=3}^{n+2}(-1)^k f\big(x^{1k}\big)$$

and

$$\begin{aligned}\iota(x_2)\theta(x_1)f(x_3,\dots,x_{n+2}) &= \theta(x_1)f(x_2,\dots,x_{n+2})\\ &= \rho(x_1)f(x_2,\dots,x_{n+2}) - \sum_{k=2}^{n+2}(-1)^k f\big(x^{1k}\big).\end{aligned}$$

We have

$$\big[\theta(x_1),\iota(x_2)\big]f(x_3,\dots,x_{n+2}) = f\big(x^{12}\big) = \iota\big([x_1,x_2]\big)f(x_3,\dots,x_{n+2}). \quad \square$$

Proposition 2.5.8

The element θ is a representation of the Lie algebra L on the vector space $\hom(\bigwedge L, V)$.

Proof. It is sufficient to show that

$$\theta\big([x_1,x_2]\big)f = \big[\theta(x_1),\theta(x_2)\big]f$$

for $\forall f$ of $\hom(\bigwedge^n L, V)$. The proof is by induction on n.

Suppose that f is an element of $\hom(\bigwedge^0 L, V)$. We have

$$\begin{aligned}\theta\big([x_1,x_2]\big)f &= \rho\big([x_1,x_2]\big)f\\ &= \big[\rho(x_1),\rho(x_2)\big]f\\ &= \big[\theta(x_1),\theta(x_2)\big]f.\end{aligned}$$

Suppose that $n>0$ and let f be an element of $\hom(\bigwedge^n L, V)$. We have

$$\iota(x_3)\theta\big([x_1,x_2]\big) = \theta\big([x_1,x_2]\big)\iota(x_3) - \iota\Big(\big[[x_1,x_2],x_3\big]\Big)$$

and

$$\begin{aligned}\iota(x_3)\theta\big([x_1,x_2]\big)f &= \theta\big([x_1,x_2]\big)\iota(x_3)f - \iota\Big(\big[[x_1,x_2],x_3\big]\Big)f\\ &= \big[\theta(x_1),\theta(x_2)\big]\iota(x_3)f - \iota\Big(\big[[x_1,x_2],x_3\big]\Big)f\end{aligned}$$

by the induction hypothesis. We have

$$\begin{aligned}\theta(x_1)\theta(x_2)\iota(x_3) &= \theta(x_1)\Big(\iota\big([x_2,x_3]\big)+\iota(x_3)\theta(x_2)\Big)\\ &= \iota\Big(\big[x_1,[x_2,x_3]\big]\Big)+\iota\big([x_2,x_3]\big)\theta(x_1)+\Big(\iota\big([x_1,x_3]\big)+\iota(x_3)\theta(x_1)\Big)\theta(x_2)\\ &= \iota\Big(\big[x_1,[x_2,x_3]\big]\Big)+\iota\big([x_2,x_3]\big)\theta(x_1)+\iota\big([x_1,x_3]\big)\theta(x_2)+\iota(x_3)\theta(x_1)\theta(x_2)\end{aligned}$$

and

$$\big[\theta(x_1),\theta(x_2)\big]\iota(x_3) = \iota\Big(\big[x_1,[x_2,x_3]\big]-\big[x_2,[x_1,x_3]\big]\Big)+\iota(x_3)\big[\theta(x_1),\theta(x_2)\big].$$

We have

$$\begin{aligned}\iota(x_3)\theta\big([x_1,x_2]\big)f = \iota\Big(\big[x_1,[x_2,x_3]\big] - \big[x_2,[x_1,x_3]\big] - \big[[x_1,x_2],x_3\big]\Big)f \\ + \iota(x_3)\big[\theta(x_1),\theta(x_2)\big]f = \iota(x_3)\big[\theta(x_1),\theta(x_2)\big]f\end{aligned}$$

and

$$\theta\big([x_1,x_2]\big)f = \big[\theta(x_1),\theta(x_2)\big]f$$

since the element x_3 is arbitrary. □

PROPOSITION 2.5.9
The element $\theta(x)$ commutes with the element ∂ for $\forall x$.

Proof. It is sufficient to show that

$$\big[\partial,\theta(x)\big]f = 0$$

for $\forall f$ of $\hom(\bigwedge^n L, V)$. The proof is by induction on n. Suppose that f is an element of $\hom(\bigwedge^0 L, V)$. We have

$$\big[\partial,\theta(x)\big]f(x_1) = \rho(x_1)\rho(x)f - \rho(x)\rho(x_1)f + \rho\big([x,x_1]\big)f = 0.$$

Suppose that $n > 0$ and let f be an element of $\hom(\bigwedge^n L, V)$. We have

$$\begin{aligned}\iota(x_1)\partial\theta(x) &= \big(\theta(x_1) - \partial\iota(x_1)\big)\theta(x) \\ &= \theta(x_1)\theta(x) - \partial\Big(\theta(x)\iota(x_1) - \iota\big([x,x_1]\big)\Big) \\ &= \theta(x_1)\theta(x) - \partial\theta(x)\iota(x_1) + \partial\iota\big([x,x_1]\big)\end{aligned}$$

and

$$\begin{aligned}\iota(x_1)\theta(x)\partial &= \Big(\theta(x)\iota(x_1) - \iota\big([x,x_1]\big)\Big)\partial \\ &= \theta(x)\big(\theta(x_1) - \partial\iota(x_1)\big) - \iota\big([x,x_1]\big)\partial \\ &= \theta(x)\theta(x_1) - \theta(x)\partial\iota(x_1) + \partial\iota\big([x,x_1]\big) - \theta\big([x,x_1]\big) \\ &= \theta(x_1)\theta(x) - \theta(x)\partial\iota(x_1) + \partial\iota\big([x,x_1]\big).\end{aligned}$$

We have $\iota(x_1)\big[\partial,\theta(x)\big] = -\big[\partial,\theta(x)\big]\iota(x_1)$ and

$$\iota(x_1)\big[\partial,\theta(x)\big]f = -\big[\partial,\theta(x)\big]\iota(x_1)f = 0$$

by the induction hypothesis. We have

$$\big[\partial,\theta(x)\big]f = 0$$

since the element x_1 is arbitrary. □

THEOREM 2.5.2
We have $\partial^2 = 0$.

Proof. It is sufficient to show that $\partial^2 f = 0$ for $\forall f$ of $\hom(\bigwedge^n L, V)$. The proof is by induction on n. Suppose that f is an element of $\hom(\bigwedge^0 L, V)$. We have

$$\begin{aligned}\partial^2 f(x_1, x_2) &= \rho(x_1)\partial f(x_2) - \rho(x_2)\partial f(x_1) - \partial f\big([x_1, x_2]\big) \\ &= \rho(x_1)\rho(x_2)f - \rho(x_2)\rho(x_1)f - \rho\big([x_1, x_2]\big)f = 0.\end{aligned}$$

Suppose that $n > 0$ and let f be an element of $\hom(\bigwedge^n L, V)$. We have

$$\begin{aligned}\iota(x)\partial^2 &= \big(\theta(x) - \partial\iota(x)\big)\partial \\ &= \theta(x)\partial - \partial\big(\theta(x) - \partial\iota(x)\big) \\ &= \partial^2\iota(x)\end{aligned}$$

and $\iota(x)\partial^2 f = \partial^2\iota(x)f = 0$ by the induction hypothesis. We have $\partial^2 f = 0$ since the element x is arbitrary. $\square$

2.6 Weyl's Theorem

DEFINITION 2.6.1
A representation of a Lia algebra on a vector space $V \neq \{0\}$ is said to be irreducible if the subspaces V and $\{0\}$ are the only invariant subspaces.

DEFINITION 2.6.2
A Lie algebra is said to be simple if the adjoint representation is irreducible.

PROPOSITION 2.6.1
A simple Lie algebra is either a semisimple Lie algebra or a commutative Lie algebra of dimension one.

Proof. A simple Lie algebra of dimension greater than one is not commutative. $\square$

PROPOSITION 2.6.2
The derived Lie algebra of a semisimple simple Lie algebra is itself.

PROPOSITION 2.6.3
Suppose that L is a finite dimensional semisimple Lie algebra over a field of characteristic 0 and let L_0 be an ideal of L. We have $L = L_0 \oplus L_0^\perp$.

Proof. Suppose that $\big\{e_k\big\}_{k=1}^{\dim L}$ is a basis of the Lie algebra L such that $\big\{e_k\big\}_{k=1}^{\dim L_0}$ is a basis of the ideal L_0. There exists a basis $\big\{f_k\big\}_{k=1}^{\dim L}$ such that

$$\big(B(e_i, f_j)\big)_{i,j=1}^{\dim L} = 1.$$

The set $\big\{f_k\big\}_{k=\dim L_0+1}^{\dim L}$ is a basis of the ideal $L_0^\perp$. We have $L = L_0 \oplus L_0^\perp$ since

$$\big[L_0, L_0^\perp\big] = L_0 \cap L_0^\perp = \{0\}. \quad \square$$

Theorem 2.6.1

A finite dimensional semisimple Lie algebra over a field of characteristic 0 is a direct sum of semisimple simple Lie algebras.

Proof. The proof is by induction on the dimension of the Lie algebra L. Suppose that $\dim L > 0$. We may assume that the Lie algebra L is not simple. There exists an ideal L_0 of the Lie algebra L such that $0 < \dim L_0 < \dim L$. The semisimple ideals L_0 and $L_0^{\perp}$ are direct sums of simple Lie algebras by the induction hypothesis. □

Corollary 2.6.1

The derived Lie algebra of a finite dimensional semisimple Lie algebra over a field of characteristic 0 is itself.

Theorem 2.6.2

Suppose that $L = \bigoplus_i L_i$ is a direct sum of semisimple simple Lie algebras over a field. We have

$$\{\, L_i : i \,\} = \{\text{a simple ideal of } L\}$$

and

$$\{\text{a partial direct sum of } \textstyle\bigoplus_i L_i\} = \{\text{an ideal of } L\}.$$

Proof. By Remark of 491 (cf. Lie Algebras). □

Corollary 2.6.2

A direct sum of semisimple simple Lie algebras over a field is semisimple.

Corollary 2.6.3

A direct sum of finite dimensional semisimple Lie algebras over a field of characteristic 0 is semisimple.

Definition 2.6.3

A finite dimensional representation of a Lie algebra is said to be completely reducible if the space is an internal direct sum of irreducible invariant subspaces.

Proposition 2.6.4

A representation of a Lie algebra on a finite dimensional vector space $V \neq \{0\}$ is associated with an irreducible invariant subspace.

Proof. There exists an invariant subspace $V_0 \neq \{0\}$ such that

$$\dim V_0 = \min\{\, \dim V_0 : V_0 \neq \{0\} \text{ is an invariant subspace} \,\}$$

and the invariant subspace V_0 is irreducible. □

Proposition 2.6.5

The following are equivalent for a representation of a Lie algebra on a finite dimensional vector space V.

1. The representation is completely reducible.

2. Suppose that V_0 is an invariant subspace. There exists an invariant subspace $V_0^\perp$ such that $V = V_0 \oplus V_0^\perp$.

Proof. Suppose that the representation is completely reducible and let $V = \bigoplus_{k=1}^n V_k$ be an internal direct sum of irreducible invariant subspaces. The proof is by induction on $\operatorname{codim} V_0 = \dim V - \dim V_0$. Suppose that $\operatorname{codim} V_0 > 0$. There exists k such that the irreducible invariant subspace V_k is not contained in the invariant subspace V_0. We have

$$V_0 \cap V_k = \{0\}$$

since the subspace $V_0 \cap V_k \neq V_k$ is invariant. We have

$$\operatorname{codim}(V_0 \oplus V_k) < \operatorname{codim} V_0.$$

There exists an invariant subspace $(V_0 \oplus V_k)^\perp$ such that

$$\begin{aligned} V &= V_0 \oplus V_k \oplus (V_0 \oplus V_k)^\perp \\ &= V_0 \oplus \big(V_k \oplus (V_0 \oplus V_k)^\perp\big) \end{aligned}$$

by the induction hypothesis.

Suppose that the condition 2 holds and let $V_0 = \bigoplus_{k=1}^n V_k$ be an arbitrary internal direct sum of irreducible invariant subspaces. There exists an invariant subspace $V_0^\perp$ such that $V = V_0 \oplus V_0^\perp$. There exists an irreducible invariant subspace V_{n+1} of $V_0^\perp$ provided that $V_0 \neq V$ by Proposition 2.6.4. The space $\bigoplus_{k=1}^{n+1} V_k \neq V_0$ is an internal direct sum of irreducible invariant subspaces. □

Definition 2.6.4

A finite dimensional Lie algebra is said to be reductive if the adjoint representation is completely reducible.

Proposition 2.6.6

An ideal of an ideal of a reductive Lie algebra is an ideal of the Lie algebra.

Proof. Suppose that L_0 is an ideal of a reductive Lie algebra L and let L_1 be an ideal of the ideal L_0. There exists an ideal $L_0^\perp$ of L such that $L = L_0 \oplus L_0^\perp$. The set $[L, L_1] = [L_0 \oplus L_0^\perp, L_1] = [L_0, L_1]$ is contained in the set L_1. □

Proposition 2.6.7

A finite dimensional semisimple Lie algebra over a field of characteristic 0 is reductive.

Proof. By Proposition 2.6.3. □

Proposition 2.6.8

We have

$$\big\{\text{a simple ideal of } L\big\} = \big\{\text{an irreducible invariant subspace for } \mathrm{ad}_L\big\}$$

for a reductive Lie algebra L.

Proof. A simple ideal of the Lie algebra L is an irreducible invariant subspace for the adjoint representation of the Lie algebra L.

Suppose that L_0 is an irreducible invariant subspace for the adjoint representation of the Lie algebra L. The ideal L_0 is simple since an ideal of the ideal L_0 is an invariant subspace for the adjoint representation of the Lie algebra L by Proposition 2.6.6. □

Theorem 2.6.3

We have the following for a reductive Lie algebra L.

1. The reductive Lie algebra is an internal direct sum of the derived Lie algebra and the radical.
2. The derived Lie algebra is semisimple.
3. The radical is commutative.

Proof. We may assume that

$$L = S \oplus A, \qquad S = \bigoplus_{k=1}^{n} L_k,$$

where the ideal L_k is semisimple and simple for $\forall k$ and the ideal A is commutative by Proposition 2.6.8 and Proposition 2.6.1. The ideal S is semisimple by Corollary 2.6.2. We have $S = DS = DL$ by Proposition 2.6.2. There exists the radical of the Lie algebra L by Corollary 2.4.5. We have $DL \cap \operatorname{rad} L = \{0\}$ since it is a solvable ideal of the semisimple ideal DL. We have $A = \operatorname{rad} L$. □

Definition 2.6.5

The kernel of the adjoint representation of a Lie algebra is a commutative ideal and it is called the center.

Remark 2.6.1

The adjoint representation of a semisimple Lie algebra is faithful.

Theorem 2.6.4

A finite dimensional Lie algebra over a field of characteristic 0 is reductive if and only if it is a direct sum of a semisimple Lie algebra and a commutative Lie algebra.

Proof. Suppose that the Lie algebra $L = S \oplus A$ is a direct sum of a semisimple Lie algebra S and a commutative Lie algebra A. We have $S = DS = DL$ by Corollary 2.6.1. We have $DL \cap \operatorname{rad} L = \{0\}$ since it is a solvable ideal of the semisimple ideal DL. We have $A = \operatorname{rad} L$. The Lie algebra is reductive since it is a direct sum of simple Lie algebras by Theorem 2.6.1. □

Corollary 2.6.4

Suppose that L is a reductive Lie algebra over a field of characteristic 0.

$$L = DL \oplus \operatorname{rad} L$$

is the unique decomposition of the Lie algabra into an internal direct sum of a semisimple ideal and a commutative ideal.

Proposition 2.6.9
Suppose that L is a reductive Lie algebra over a field of characteristic 0. We have

$$\operatorname{rad} L = L^{\perp}.$$

Proof. The radical is contained in the ideal $L^{\perp}$. Suppose that x is an element of the ideal $L^{\perp}$ and let $x = (x - x_0) \oplus x_0$ be the decomposition in the direct sum $L = DL \oplus \operatorname{rad} L$. We have $x = x_0$ since the Killing form of the derived Lie algebra is nondegenerate and we have $B(x - x_0, DL) = B(x, DL) = \{0\}$. □

Example 2.6.1
Suppose that $\mathbb{F}$ is a field of characteristic 0. The Lie algebra

$$M(m, \mathbb{F}) = \left\{ x \in M(m, \mathbb{F}) : \operatorname{tr} x = 0 \right\} \oplus \mathbb{F}$$

is reductive and we have

$$DM(m, \mathbb{F}) = \left\{ x \in M(m, \mathbb{F}) : \operatorname{tr} x = 0 \right\}, \qquad M(m, \mathbb{F})^{\perp} = \mathbb{F}$$

for $m \geq 1$.

Proposition 2.6.10
Suppose that L_0 is an ideal of a Lie algebra L and let π be the canonical homomorphism of the Lie algebra L onto the quotient Lie algebra L/L_0. We have

$$\begin{aligned}\{\text{an ideal of } L \text{ containing } L_0\} &= \Big\{\text{an ideal } L_1 \text{ of } L \text{ such that } \pi^{-1}\big(\pi(L_1)\big) = L_1\Big\} \\ &= \{\text{an ideal of } L/L_0\}.\end{aligned}$$

Proposition 2.6.11
Suppose that L_0 is an ideal of a Lie algebra L and let π be the canonical homomorphism of the Lie algebra L onto the quotient Lie algebra L/L_0. We have

$$L/L_1 = (L/L_0)/\pi(L_1)$$

for an ideal L_1 of the Lie algebra L containing the ideal L_0.

Proposition 2.6.12
Suppose that L is a finite dimensional Lie algebra such that $L^{\perp} \cap DL = \{0\}$. We have the following.

1. The ideal $L^{\perp}$ is the largest commutative ideal of the Lie algebra L.
2. The quotient Lie algebra $L/L^{\perp}$ is semisimple.

Proof. 1. We have $\left[L^{\perp}, L^{\perp}\right] = L^{\perp} \cap DL = \{0\}$.

2. We write π for the canonical homomorphism of the Lie algebra L onto the quotient Lie algebra $L/L^\perp$ and let L_0 be an ideal of the Lie algebra L containing $L^\perp$ such that the ideal $\pi(L_0)$ is commutative. We have

$$\bigl[L_0, L_0\bigr] = L^\perp \cap DL = \{0\}$$

since we have $\pi\Bigl(\bigl[L_0, L_0\bigr]\Bigr) = \bigl[\pi(L_0), \pi(L_0)\bigr] = \{0\}$. □

THEOREM 2.6.5
A finite dimensional Lie algebra over a field of characteristic 0 is reductive if and only if we have $L^\perp \cap DL = \{0\}$.

Proof. We write π for the canonical homomorphism of the Lie algebra L onto the quotient Lie algebra $L/L^\perp$. Suppose that we have $L^\perp \cap DL = \{0\}$. We have

$$L = L^\perp \oplus DL$$

since we have

$$\begin{aligned}\pi(DL) &= D\bigl(\pi(L)\bigr)\\ &= D(L/L^\perp) = L/L^\perp.\end{aligned}$$ □

PROPOSITION 2.6.13
Suppose that R is a commutative ring with identity and let M be a unital module over R. We have $\hom(R, M) = M$.

PROPOSITION 2.6.14
Suppose that

$$\rho_1 \colon L \to \hom V_1, \qquad \rho_2 \colon L \to \hom V_2$$

are representations of a Lie algebra L.

$$\rho(x)f = \rho_2(x) \circ f - f \circ \rho_1(x)$$

defines a representation ρ of the Lie algebra L on the vector space $\hom(V_1, V_2)$.

Proof. We have

$$\begin{aligned}\rho(x_1)\rho(x_2)f = \rho_2(x_1) \circ \bigl(\rho_2(x_2) \circ f - f \circ \rho_1(x_2)\bigr)&\\ - \bigl(\rho_2(x_2) \circ f - f \circ \rho_1(x_2)\bigr) \circ \rho_1(x_1)&\end{aligned}$$

and

$$\begin{aligned}\bigl[\rho(x_1), \rho(x_2)\bigr]f = \bigl(\rho_2(x_1) \circ \rho_2(x_2) - \rho_2(x_2) \circ \rho_2(x_1)\bigr) \circ f&\\ - f \circ \bigl(\rho_1(x_1) \circ \rho_1(x_2) - \rho_1(x_2) \circ \rho_1(x_1)\bigr).&\end{aligned}$$

We have

$$\begin{aligned}\bigl[\rho(x_1), \rho(x_2)\bigr]f &= \bigl[\rho_2(x_1), \rho_2(x_2)\bigr] \circ f - f \circ \bigl[\rho_1(x_1), \rho_1(x_2)\bigr]\\ &= \rho_2\bigl([x_1, x_2]\bigr) \circ f - f \circ \rho_1\bigl([x_1, x_2]\bigr)\\ &= \rho\bigl([x_1, x_2]\bigr)f.\end{aligned}$$ □

Weyl's Theorem

Any finite dimensional representation of a finite dimensional semisimple Lie algebra over a field of characteristic 0 is completely reducible.

Theorem 2.6.6

Suppose that ρ is a finite dimensional representation of a finite dimensional semisimple Lie algebra over a field of characteristic 0. We have $H^1(\rho) = \{0\}$.

Proof of Weyl's theorem. Suppose that L is a finite dimensional semisimple Lie algebra over a field of characteristic 0 and let ρ be a representation of the Lie algebra L on a finite dimensional vector space V.

Suppose that V_0 is an invariant subspace for ρ. There exists a subspace V_1' such that $V = V_0 \oplus V_1'$. There exists an element f of $\hom\bigl(L, \hom(V_1', V_0)\bigr)$ such that

$$\rho(x) = \begin{pmatrix} \rho_0(x) & f(x) \\ 0 & \rho_1'(x) \end{pmatrix}$$

for $\forall x$. We have

$$f\bigl([x_1, x_2]\bigr) = \rho_0(x_1)f(x_2) + f(x_1)\rho_1'(x_2) - \rho_0(x_2)f(x_1) - f(x_2)\rho_1'(x_1)$$

for $\forall(x_1, x_2)$ since we have $\rho\bigl([x_1, x_2]\bigr) = \bigl[\rho(x_1), \rho(x_2)\bigr]$. The representations ρ_0 and ρ_1' are associated with a representation on the vector space $\hom(V_1', V_0)$ by Proposition 2.6.14 and we have $\partial f(x_1, x_2) = 0$ for $\forall(x_1, x_2)$. There exists an element f_0 of $\hom(V_1', V_0)$ such that

$$\begin{aligned} f(x) &= -\partial f_0(x) \\ &= -\rho_0(x) f_0 + f_0 \rho_1'(x) \end{aligned}$$

for $\forall x$ since we have $H^1\bigl(L, \hom(V_1', V_0)\bigr) = \{0\}$ by Theorem 2.6.6. We define

$$V_1 = \begin{pmatrix} 1 & f_0 \\ 0 & 1 \end{pmatrix} V_1'.$$

We have $V = V_0 \oplus V_1$. We have

$$\begin{aligned} \rho(x) \begin{pmatrix} 1 & f_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ v_1' \end{pmatrix} &= \begin{pmatrix} \rho_0(x) & f(x) \\ 0 & \rho_1'(x) \end{pmatrix} \begin{pmatrix} 1 & f_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ v_1' \end{pmatrix} \\ &= \begin{pmatrix} \rho_0(x) & f(x) \\ 0 & \rho_1'(x) \end{pmatrix} \begin{pmatrix} f_0 v_1' \\ v_1' \end{pmatrix} \\ &= \begin{pmatrix} \rho_0(x) f_0 v_1' + f(x) v_1' \\ \rho_1'(x) v_1' \end{pmatrix} \\ &= \begin{pmatrix} f_0 \rho_1'(x) v_1' \\ \rho_1'(x) v_1' \end{pmatrix} \\ &= \begin{pmatrix} 1 & f_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \rho_1'(x) v_1' \end{pmatrix} \end{aligned}$$

for $\forall(x, v_1')$. $\square$

Lemma 2.6.1

Suppose that V is a finite dimensional vector space over a field of characteristic 0 and let L be a semisimple irreducible Lie subalgebra of $\hom V$. We have $H^1(L, V) = \{0\}$.

Proof of Theorem 2.6.6. Suppose that L is a finite dimensional semisimple Lie algebra over a field of characteristic 0 and let ρ be an irreducible representation of the Lie algebra on a finite dimensional vector space V. We define

$$\bar{\rho}(x + \ker \rho) = \rho(x)$$

for $\forall x$. The Lie algebra $L/\ker \rho = (\ker \rho)^\perp$ is semisimple. We have $H^1(\bar{\rho}) = \{0\}$ by Lemma 2.6.1. Suppose that ϕ is an element of $\hom(L, V)$ such that $\partial\phi = 0$. We have

$$\partial\phi(x_1, x_2) = \rho(x_1)\phi(x_2) - \rho(x_2)\phi(x_1) - \phi\big([x_1, x_2]\big) = 0$$

for $\forall(x_1, x_2)$. The subspace $\ker \rho = D \ker \rho$ is contained in the subspace $\ker \phi$. We define

$$\bar{\phi}(x + \ker \rho) = \phi(x)$$

for $\forall x$. We have

$$\partial\bar{\phi}(x_1 + \ker \rho, x_2 + \ker \rho) = \partial\phi(x_1, x_2) = 0$$

for $\forall(x_1, x_2)$. There exists an element v of the vector space V such that $\bar{\phi} = \partial v$ since we have $H^1(\bar{\rho}) = \{0\}$. We have

$$\begin{aligned} \phi(x) &= \bar{\phi}(x + \ker \rho) \\ &= \partial v(x + \ker \rho) \\ &= \bar{\rho}(x + \ker \rho)v \\ &= \rho(x)v \\ &= \partial v(x) \end{aligned}$$

for $\forall x$. We have $H^1(\rho) = \{0\}$.

The proof is by induction on $\dim V$. Suppose that $\dim V > 0$. We may assume that the representation ρ is reducible. There exists an invariant subspace V_0 for the representation ρ such that $0 < \dim V_0 < \dim V$. We have $H^1(\rho_{V_0}) = \{0\}$ and $H^1(\rho_{V/V_0}) = \{0\}$ by the induction hypothesis. Suppose that ϕ is an element of $\hom(L, V)$ such that $\partial\phi = 0$. We have

$$\partial\phi(x_1, x_2) = \rho(x_1)\phi(x_2) - \rho(x_2)\phi(x_1) - \phi\big([x_1, x_2]\big) = 0$$

for $\forall(x_1, x_2)$. We define an element ϕ_{V/V_0} of $\hom(L, V/V_0)$ by

$$\phi_{V/V_0}(x) = \phi(x) + V_0$$

for $\forall x$. We have

$$\partial\phi_{V/V_0}(x_1, x_2) = \partial\phi(x_1, x_2) + V_0 = 0$$

for $\forall(x_1, x_2)$. There exists an element $v_1 + V_0$ of the vector space V/V_0 such that $\phi_{V/V_0} = \partial(v_1 + V_0)$ since we have $H^1(\rho_{V/V_0}) = \{0\}$. We have

$$\begin{aligned}\phi(x) + V_0 &= \phi_{V/V_0}(x)\\ &= \partial(v_1 + V_0)(x)\\ &= \rho_{V/V_0}(x)(v_1 + V_0)\\ &= \rho(x)v_1 + V_0\end{aligned}$$

for $\forall x$. We define an element ϕ_{V_0} of $\hom(L, V_0)$ by

$$\phi_{V_0}(x) = \phi(x) - \rho(x)v_1$$

for $\forall x$. We have

$$\partial\phi_{V_0}(x_1, x_2) = \partial\phi(x_1, x_2) - \Big(\rho(x_1)\rho(x_2) - \rho(x_2)\rho(x_1) - \rho\big([x_1, x_2]\big)\Big)v_1 = 0$$

for $\forall(x_1, x_2)$. There exists an element v_0 of the vector space V_0 such that $\phi_{V_0} = \partial v_0$ since we have $H^1(\rho_{V_0}) = \{0\}$. We have

$$\begin{aligned}\phi(x) &= \phi_{V_0}(x) + \rho(x)v_1\\ &= \partial v_0(x) + \rho(x)v_1\\ &= \rho(x)(v_0 + v_1)\\ &= \partial(v_0 + v_1)(x)\end{aligned}$$

for $\forall x$. □

LEMMA 2.6.2 (Schur)
Suppose that ρ is an irreducible representation of a Lie algebra on a vector space V. An element C of $\hom V \setminus \{0\}$ is an epimorphism if $\big[\rho(x), C\big] = 0$ for $\forall x$.

Proof. The subspace $CV \neq \{0\}$ is invariant for ρ. □

Suppose that B is a bilinear form on a finite dimensional vector space V.

PROPOSITION 2.6.15
We have the following provided that B is nondegenerate.

1. There exist bases $\big\{e_k\big\}_{k=1}^n$ and $\big\{f_k\big\}_{k=1}^n$ such that $\big(B(e_i, f_j)\big)_{i,j=1}^n = 1$.

2. The element $\sum_{k=1}^n e_k \otimes f_k$ is independent of the choice of the bases.

Proof. 1. Suppose that $\big\{e_k\big\}_{k=1}^n$ is any basis. The matrix $\big(B(e_i, e_j)\big)_{i,j=1}^n$ is invertible and let $(e_1, \dots, e_n) = (f_1, \dots, f_n)\big(B(e_i, e_j)\big)_{i,j=1}^n$. We have

$$\big(B(e_i, f_j)\big)_{i,j=1}^n = \big(B(e_i, e_j)\big)_{i,j=1}^n\Big(\big(B(e_i, e_j)\big)_{i,j=1}^n\Big)^{-1} = 1$$

by Remark of 489 (cf. Algebra).

2. Suppose that

$$(v_1, \dots, v_n) = (e_1, \dots, e_n)P, \qquad (w_1, \dots, w_n) = (f_1, \dots, f_n)Q$$

are arbitrary bases such that $\big(B(v_i, w_j)\big)_{i,j=1}^n = 1$. We have

$$1 = P^T Q$$

by Remark of 489 (cf. Algebra). We have

$$\begin{aligned}\sum_{k=1}^n v_k \otimes w_k &= \sum_{j=1}^n \Big(\sum_{i=1}^n P_{ij} e_i\Big) \otimes w_j \\ &= \sum_{i=1}^n e_i \otimes \Big(\sum_{j=1}^n P_{ij} w_j\Big) \\ &= \sum_{i=1}^n e_i \otimes \Big(\sum_{j=1}^n Q^{-1}_{ji} w_j\Big) \\ &= \sum_{k=1}^n e_k \otimes f_k. \end{aligned}$$

□

Proposition 2.6.16
The bilinear form B is nondegenerate if and only if the linear mapping

$$x \mapsto \big[y \mapsto B(x,y)\big] \tag{2.6}$$

is an isomorphism of V onto V^*.

Proof. Suppose that B is nondegenerate. We write $\{\delta^k\}_{k=1}^n$ for the dual basis of the basis $\{f_k\}_{k=1}^n$. The linear mapping (2.6) is isomorphism since

$$B(e_k, x) = \delta^k(x)$$

for $\forall x$ and for $\forall k$.

Suppose that the linear mapping (2.6) is an isomorphism. Suppose that $\{f_k\}_{k=1}^n$ is an arbitrary basis and let $\{\delta^k\}_{k=1}^n$ be the dual basis. There exists a basis $\{e_k\}_{k=1}^n$ such that

$$B(e_k, x) = \delta^k(x)$$

for $\forall x$ and for $\forall k$. We have $\big(B(e_i, f_j)\big)_{i,j=1}^n = 1$. □

Suppose that V is a finite dimensional vector space over a field of characteristic 0 and let L be a semisimple Lie subalgebra of $\hom V$.

Proposition 2.6.17
An invariant symmetric form on L

$$(x_1, x_2) \mapsto \operatorname{tr}(x_1 x_2)$$

is nondegenerate.

Proof. The ideal $L^\perp = \{x \in L : \mathrm{tr}(xL) = \{0\}\}$ is solvable by Chevalley's theorem since $\mathrm{tr}(L^\perp \cdot DL^\perp) = \mathrm{tr}(L^\perp \cdot L) = \{0\}$. We have $L^\perp = \{0\}$. □

There exist bases $\{e_k\}_{k=1}^n$ and $\{f_k\}_{k=1}^n$ such that $\big(\mathrm{tr}(e_i f_j)\big)_{i,j=1}^n = 1$ by Proposition 2.6.15. We write

$$(\mathrm{ad}\, x)(e_1, \ldots, e_n) = (e_1, \ldots, e_n)\alpha(x), \quad (\mathrm{ad}\, x)(f_1, \ldots, f_n) = (f_1, \ldots, f_n)\beta(x)$$

for $\forall x$.

Proposition 2.6.18
We have the following.

1. We have $\alpha(x) = \Big(\mathrm{tr}\big((\mathrm{ad}\, x)(e_j) f_i\big)\Big)_{i,j=1}^n$ for $\forall x$.

2. We have $\alpha(x) + \beta(x)^T = 0$ for $\forall x$.

Proof. 1. We have $\mathrm{tr}\big((\mathrm{ad}\, x)(e_j) f_i\big) = \alpha_{ij}(x)$.

2. We have $\alpha_{ij}(x) + \beta_{ji}(x) = \mathrm{tr}\big((\mathrm{ad}\, x)(e_j) f_i\big) + \mathrm{tr}\big(e_j (\mathrm{ad}\, x)(f_i)\big) = 0$. □

We define

$$C = \sum_{k=1}^n e_k f_k.$$

Proposition 2.6.19
We have $[x, C] = 0$ for $\forall x$ of L.

Proof. We have

$$\begin{aligned}[x, C] &= \sum_{k=1}^n \big((\mathrm{ad}\, x)(e_k) f_k + e_k (\mathrm{ad}\, x)(f_k)\big) \\ &= \sum_{i,j=1}^n \big(\alpha_{ij}(x) e_i f_j + e_i \beta_{ji}(x) f_j\big) = 0.\end{aligned}$$ □

Proof of Lemma 2.6.1. We may assume that $L \neq \{0\}$. We have $\mathrm{tr}\, C = \dim L \neq 0$ since the underlying field is of characteristic 0. The element C is invertible by Schur's Lemma.

Suppose that ϕ is an element of $\hom(L, V)$ such that $\partial\phi = 0$. We have

$$\partial\phi(x_1, x_2) = x_1\phi(x_2) - x_2\phi(x_1) - \phi\big([x_1, x_2]\big) = 0$$

for $\forall (x_1, x_2)$. We define

$$v = C^{-1} \sum_{k=1}^n e_k \phi(f_k).$$

Suppose that x is an arbitrary element of the Lie subalgebra L. We have

$$\begin{aligned}\partial v(x) &= xv \\ &= C^{-1}x\sum_{k=1}^{n} e_k\phi(f_k) \\ &= C^{-1}\Big(\sum_{k=1}^{n}(\operatorname{ad} x)(e_k)\phi(f_k) + \sum_{k=1}^{n} e_k x\phi(f_k)\Big),\end{aligned}$$

where

$$\begin{aligned}\sum_{k=1}^{n}(\operatorname{ad} x)(e_k)\phi(f_k) &= \sum_{i,j=1}^{n}\alpha_{ij}(x)e_i\phi(f_j) \\ &= -\sum_{i,j=1}^{n}\beta_{ji}(x)e_i\phi(f_j) \\ &= -\sum_{k=1}^{n} e_k\phi\big((\operatorname{ad} x)(f_k)\big).\end{aligned}$$

We have

$$\begin{aligned}\partial v(x) &= C^{-1}\sum_{k=1}^{n} e_k\Big(-\phi\big((\operatorname{ad} x)(f_k)\big) + x\phi(f_k)\Big) \\ &= C^{-1}\sum_{k=1}^{n} e_k f_k\phi(x) = \phi(x).\end{aligned}$$ □

Proposition 2.6.20
Suppose that L is a finite dimensional semisimple Lie algebra over a field of characteristic 0 and let ρ be a finite dimensional representation of L. We have $\operatorname{tr}\rho(L) = \{0\}$.

Proof. We have

$$\begin{aligned}\operatorname{tr}\rho(L) &= \operatorname{tr}\rho(DL) \\ &= \operatorname{span}\operatorname{tr}\big[\rho(L), \rho(L)\big] = \{0\}\end{aligned}$$

since we have $L = DL$ by Corollary 2.6.1. □

Lemma 2.6.3 (Schur)
Suppose that ρ is an irreducible representation of a Lie algebra over an algebraically closed field on a finite dimensional vector space V and let C be an element of $\hom V$. The element C is scalar if we have $\big[\rho(x), C\big] = 0$ for $\forall x$.

Proof. We may assume that $V \neq \{0\}$. There exists an eigenvalue ν of the element C since the underlying field is algebraically closed. The subspace $(C - \nu)V \neq V$ is invariant for the irreducible representation ρ. □

Theorem 2.6.7
Suppose that V is a finite dimensional vector space over an algebraically closed field of characteristic 0 and let L be a semisimple Lie subalgebra of $\hom V$.

1. Suppose that
$$\bar{x} = S + N$$
is the Jordan decomposition of an element of the Lie subalgebra L. The elements S and N belong to the Lie subalgebra L.

2. Suppose that ρ is a representation of the Lie subalgebra L on a finite dimensional vector space.
$$\rho(\bar{x}) = \rho(S) + \rho(N)$$
is the Jordan decomposition.

Proof. 1. The restriction of the adjoint representation ad of the Lie algebra $\hom V$ to the Lie subalgebra L is completely reducible by Weyl's theorem. There exists an invariant subspace $L^{\perp}$ for $\operatorname{ad} L$ such that we have $\hom V = L \oplus L^{\perp}$. The set $\left[L, L^{\perp}\right]$ is contained in the set $L^{\perp}$. The subspaces L and $L^{\perp}$ are invariant for the element $\operatorname{ad} S$ by Corollary 2.3.2 since
$$\operatorname{ad} \bar{x} = \operatorname{ad} S + \operatorname{ad} N$$
is the Jordan decomposition by Corollary 2.4.1. There exists a unique element $S^{\perp}$ of $L^{\perp}$ such that the element $S - S^{\perp}$ belongs to the Lie subalgebra L. We have
$$\left[S^{\perp}, L\right] = \left[S - (S - S^{\perp}), L\right] = L \cap L^{\perp} = \{0\}.$$
The Lie subalgebra L is completely reducible by Weyl's theorem and let
$$V = \bigoplus_{k=1}^{n} V_k$$
be a decomposition into an internal direct sum of irreducible invariant subspaces for the Lie subalgebra L. Suppose that
$$\bar{x}_k = S_k + N_k$$
is the Jordan decomposition for $\forall k$. We have
$$S = \operatorname{diag}(S_1, \ldots, S_n), \qquad N = \operatorname{diag}(N_1, \ldots, N_n).$$
We have
$$\begin{aligned} 0 &= \operatorname{tr} \bar{x}_k \\ &= \operatorname{tr} S_k = \operatorname{tr} S_k^{\perp} \end{aligned}$$

for $\forall k$ by Proposition 2.6.20. The element $S_k^{\perp}$ is scalar for $\forall k$ by Schur's Lemma since we have

$$0 = \big[x, S^{\perp}\big] = \operatorname{diag}\Big(\big[x_1, S_1^{\perp}\big], \ldots, \big[x_n, S_n^{\perp}\big]\Big)$$

for $\forall x$ of the Lie subalgebra L. We have $S^{\perp} = 0$.

2. We write ad for the adjoint representations of the Lie subalgebras L and $\rho(L)$.

$$\operatorname{ad} \bar{x} = \operatorname{ad} S + \operatorname{ad} N$$

is the Jordan decomposition by Corollary 2.4.1 and Corollary 2.3.4. The element $\operatorname{ad} \rho(S)$ is semisimple since we have

$$L = \bigoplus_{\nu} \ker(\operatorname{ad} S - \nu)$$

by Proposition 2.3.3 and we have

$$\begin{aligned} \rho(L) &= \sum_{\nu} \rho\big(\ker(\operatorname{ad} S - \nu)\big) \\ &= \sum_{\nu} \ker\big(\operatorname{ad} \rho(S) - \nu\big). \end{aligned}$$

The element $\operatorname{ad} \rho(N)$ is nilpotent since the element $\operatorname{ad} N$ is nilpotent.

$$\operatorname{ad} \rho(\bar{x}) = \operatorname{ad} \rho(S) + \operatorname{ad} \rho(N)$$

is the Jordan decomposition. The Lie subalgebra $\rho(L) = L/\ker\rho$ is semisimple by Theorem 2.6.2 and

$$\rho(\bar{x}) = \rho(S) + \rho(N)$$

is the Jordan decomposition since the adjoint representation of the Lie subalgebra $\rho(L)$ is faithful. □

Definition 2.6.6
An element x of a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0 is said to be semisimple (resp. nilpotent) if the element $\operatorname{ad} x$ is semisimple (resp. nilpotent).

Definition 2.6.7
A Lie subalgebra of a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0 is called a toral subalgebra if it consists of semisimple elements.

Proposition 2.6.21
A toral subalgebra of a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0 is abelian.

Proof. We write ad the adjoint representation of the Lie subalgebra L. We assume that the Lie subalgebra L is not abelian. There exist elements x_1 and x_2 of the Lie subalgebra L such that we have $[x_1, x_2] \neq 0$. We have

$$\begin{aligned} L &= \bigoplus_\nu \ker(\operatorname{ad} x_1 - \nu) \\ &= \bigoplus_\nu \ker(\operatorname{ad} x_2 - \nu). \end{aligned}$$

since the elements $\operatorname{ad} x_1$ and $\operatorname{ad} x_2$ are semisimple by Corollary 2.3.4. There exists an element $\nu \neq 0$ such that we have $\ker(\operatorname{ad} x_1 - \nu) \neq \{0\}$ since we have $L \neq \ker(\operatorname{ad} x_1)$. We may assume that $[x_1, x_2] = \nu x_2$ and let $x_1 = \bigoplus_\nu x_{1\nu}$ be the decomposition in the direct sum $\bigoplus_\nu \ker(\operatorname{ad} x_2 - \nu)$. We have

$$-\nu x_2 = [x_2, x_1] = \bigoplus_\nu \nu x_{1\nu} = 0$$

since the element $-\nu x_2$ belongs to the subspace $\ker(\operatorname{ad} x_2)$. □

2.7 Lie Groups

Proposition 2.7.1

Any two points of a connected smooth manifold belong to the image of some smooth curve of the real line into the manifold.

Proposition 2.7.2

[1] Suppose that Δ is an involutive distribution of rank m on a smooth manifold and let H be an integral manifold[2] of the distribution Δ. Suppose that $U = C^n_\varepsilon(0)$ is a flat chart[3] with respect to the distribution Δ. A component of the manifold $H \cap U$ is an open set of some slice

$$C^m_\varepsilon(0) \times \left\{ \left(x_0^{m+1}, \ldots, x_0^n \right) \right\}$$

and a regular submanifold.

Theorem 2.7.1 (Global Frobenius Theorem)

[4] The collection of all maximal connected integral manifolds of an involutive distribution on a nonempty smooth manifold forms a foliation[5].

Theorem 2.7.2

An involutive distribution on a smooth manifold induces a foliation.

[1] [4, Proposition 19.16].
[2] [4, p. 491].
[3] [4, p. 496].
[4] [4, Theorem 19.21].
[5] [4, p. 501].

Theorem 2.7.3
Suppose that f is a homomorphism of a Lie group G into a Lie group H. The following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathrm{Lie}(G) & \xrightarrow{f_*} & \mathrm{Lie}(H) \end{array}$$

Proof. Suppose that X is an element of the Lie algebra $\mathrm{Lie}(G)$. We have

$$f(\exp X) = \exp\left(\left(\frac{\partial}{\partial t} f\big(\exp(tX)\big)\right)_{t=0}\right) = \exp\big(f_*(X)\big)$$

since the mapping $t \mapsto f\big(\exp(tX)\big)$ is a one-parameter subgroup. □

Corollary 2.7.1
The exponential mapping of a Lie subgroup of a Lie group is the restriction of the exponential mapping of the Lie group.

Proposition 2.7.3
Suppose that v_1 and v_2 are smooth vector fields on a smooth manifold. We have $L_{v_1} v_2 = [v_1, v_2]$.

Proof. We write θ_1 and θ_2 for the flows generated by the smooth vector fields v_1 and v_2. We have

$$(L_{v_1} v_2)(x) = \left(\frac{\partial}{\partial t_1} \theta_1(-t_1)_* v_2\big(\theta_1(t_1)(x)\big)\right)_{t_1=0}.$$

Suppose that f is an arbitrary smooth function. We have

$$\begin{aligned}(L_{v_1} v_2)(x)(f) &= \left(\frac{\partial}{\partial t_1} v_2\big(\theta_1(t_1)(x)\big)\big(f \circ \theta_1(-t_1)\big)\right)_{t_1=0} \\ &= \left(\frac{\partial^2}{\partial t_1 \partial t_2} f \circ \theta_1(-t_1) \circ \theta_2(t_2) \circ \theta_1(t_1)(x)\right)_{(t_1,t_2)=(0,0)}\end{aligned}$$

since we have

$$\begin{aligned}\left(\frac{\partial}{\partial t_2} f \circ \theta_1(-t_1) \circ \theta_2(t_2) \circ \theta_1(t_1)(x)\right)_{t_2=0} &= \big(f \circ \theta_1(-t_1)\big)_* v_2\big(\theta_1(t_1)(x)\big) \\ &= v_2\big(\theta_1(t_1)(x)\big)\big(f \circ \theta_1(-t_1)\big).\end{aligned}$$

We have

$$\begin{aligned}(L_{v_1} v_2)(x)(f) = &\left(\frac{\partial^2}{\partial t_1 \partial t_2} f \circ \theta_1(t_3) \circ \theta_2(t_2) \circ \theta_1(t_1)(x)\right)_{(t_1,t_2,t_3)=(0,0,0)} \\ &- \left(\frac{\partial^2}{\partial t_2 \partial t_3} f \circ \theta_1(t_3) \circ \theta_2(t_2) \circ \theta_1(t_1)(x)\right)_{(t_1,t_2,t_3)=(0,0,0)}.\end{aligned}$$

We have

$$\left(\frac{\partial^2}{\partial t_1 \partial t_2} f \circ \theta_1(t_3) \circ \theta_2(t_2) \circ \theta_1(t_1)(x)\right)_{(t_1,t_2,t_3)=(0,0,0)} = \left(\frac{\partial^2}{\partial t_1 \partial t_2} f \circ \theta_2(t_2) \circ \theta_1(t_1)(x)\right)_{(t_1,t_2)=(0,0)}$$

and we have

$$\left(\frac{\partial^2}{\partial t_2 \partial t_3} f \circ \theta_1(t_3) \circ \theta_2(t_2) \circ \theta_1(t_1)(x)\right)_{(t_1,t_2,t_3)=(0,0,0)} = \left(\frac{\partial^2}{\partial t_1 \partial t_2} f \circ \theta_1(t_1) \circ \theta_2(t_2)(x)\right)_{(t_1,t_2)=(0,0)}.$$

We have

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial t_1 \partial t_2} f \circ \theta_2(t_2) \circ \theta_1(t_1)(x)\right)_{(t_1,t_2)=(0,0)} \\
&= \left(\frac{\partial}{\partial t_1} v_2\big(\theta_1(t_1)(x)\big)(f)\right)_{t_1=0} \\
&= \left(\frac{\partial}{\partial t_1} \sum_{j=1}^{d} v_2^j\big(\theta_1(t_1)(x)\big) \frac{\partial f\big(\theta_1(t_1)(x)\big)}{\partial x^j}\right)_{t_1=0} \\
&= \sum_{i,j=1}^{d} \left(\frac{\partial v_2^j(x)}{\partial x^i} v_1^i(x) \frac{\partial f(x)}{\partial x^j} + v_2^j(x) \frac{\partial^2 f(x)}{\partial x^i \partial x^j} v_1^i(x)\right) \\
&= \sum_{i=1}^{d} v_1^i(x) \frac{\partial}{\partial x^i} \sum_{j=1}^{d} v_2^j(x) \frac{\partial f(x)}{\partial x^j} = v_1(x)\big(v_2(f)\big)
\end{aligned}$$

and we have

$$(L_{v_1} v_2)(x)(f) = v_1(x)\big(v_2(f)\big) - v_2(x)\big(v_1(f)\big) = [v_1, v_2](x)(f). \quad \square$$

Theorem 2.7.4
Suppose that X_1 and X_2 are left invariant vector fields on a Lie group. The following are equivalent.

1. We have $[X_1, X_2] = 0$.
2. We have $\exp(t_1 X_1)\exp(t_2 X_2) = \exp(t_2 X_2)\exp(t_1 X_1)$ for any (t_1, t_2).
3. We have
$$\int^{t_1} X_1 \circ \int^{t_2} X_2 = \int^{t_2} X_2 \circ \int^{t_1} X_1$$
for any (t_1, t_2).

Proof. Suppose that we have $[X_1, X_2] = 0$. We show that we have

$$\int^{t_1} X_1 \circ \int^{t_2} X_2 = \int^{t_2} X_2 \circ \int^{t_1} X_1$$

for any (t_1, t_2). We write

$$\theta_1(t) = \int^{t} X_1, \qquad \theta_2(t) = \int^{t} X_2.$$

It is sufficient to show that we have

$$\frac{\partial}{\partial t_2}\theta_1(t_1) \circ \theta_2(t_2)(g) = X_2\big(\theta_1(t_1) \circ \theta_2(t_2)(g)\big).$$

We have

$$\frac{\partial}{\partial t_2}\theta_1(t_1) \circ \theta_2(t_2)(g) = \theta_1(t_1)_* \frac{\partial}{\partial t_2}\theta_2(t_2)(g) = \theta_1(t_1)_* X_2\big(\theta_2(t_2)(g)\big).$$

It is sufficient to show that we have

$$\frac{\partial}{\partial t_1}\theta_1(-t_1)_* X_2\big(\theta_1(t_1) \circ \theta_2(t_2)(g)\big) = 0.$$

We write $h = \theta_1(t_1) \circ \theta_2(t_2)(g)$. We have

$$\begin{aligned}
&\frac{\partial}{\partial t_1}\theta_1(-t_1)_* X_2\big(\theta_1(t_1) \circ \theta_2(t_2)(g)\big) \\
&\quad = \lim_{\Delta t_1 \to 0} \frac{\theta_1\big(-(t_1 + \Delta t_1)\big)_* X_2\big(\theta_1(t_1 + \Delta t_1) \circ \theta_2(t_2)(g)\big) - \theta_1(-t_1)_* X_2\big(\theta_1(t_1) \circ \theta_2(t_2)(g)\big)}{\Delta t_1} \\
&\quad = \theta_1(-t_1)_* \lim_{\Delta t_1 \to 0} \frac{\theta_1(-\Delta t_1)_* X_2\big(\theta_1(\Delta t_1)(h)\big) - X_2(h)}{\Delta t_1} \\
&\quad = \theta_1(-t_1)_* \left(\frac{\partial}{\partial t_1}\theta_1(-t_1)_* X_2\big(\theta_1(t_1)(h)\big)\right)_{t_1 = 0} \\
&\quad = \theta_1(-t_1)_* \big((L_{X_1} X_2)(h)\big) = 0.
\end{aligned}$$

□

Theorem 2.7.5

Suppose that f is a smooth function on an open interval of the real line and let x_0 be a point of the open interval. Suppose that n is a positive integer. The function on the open interval

$$x \mapsto \int_0^1 (1 - \theta)^{n-1} f^{(n)}\big(x_0 + \theta(x - x_0)\big)\, d\theta$$

is smooth and we have

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!} + \frac{(x - x_0)^n}{(n-1)!} \int_0^1 (1 - \theta)^{n-1} f^{(n)}\big(x_0 + \theta(x - x_0)\big)\, d\theta$$

for any x.

Proposition 2.7.4

Suppose that X_1 and X_2 are elements of the Lie algebra of a Lie group.

1. We have

$$\exp(X_1 + X_2) = \lim_{n\to\infty}\left(\exp\left(\frac{X_1}{n}\right)\exp\left(\frac{X_2}{n}\right)\right)^n.$$

2. We have

$$\exp(X_1 + X_2) = (\exp X_1)(\exp X_2) = (\exp X_2)(\exp X_1)$$

provided that we have $[X_1, X_2] = 0$.

Theorem 2.7.6

The following are equivalent for a subgroup of a Lie group.

1. The subgroup is closed.
2. The subgroup is a regular submanifold.
3. The subgroup is an imbedded Lie subgroup.

Proposition 2.7.5

Suppose that f and g are homomorphisms of a connected Lie group into a Lie group. We have $f = g$ provided that we have $Df(e) = Dg(e)$.

Proof. The following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{f,g} & H \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathrm{Lie}(G) & \xrightarrow{f_* = g_*} & \mathrm{Lie}(H) \end{array}$$

The homomorphisms f and g are identical on a neighborhood of the identity since the exponential mappings are local diffeomorphisms at the origins. We have $f = g$ since the connected Lie group is generated by the neighborhood of the identity. $\square$

Theorem 2.7.7

[6] Suppose that θ is a smooth right action of a Lie group G on a smooth manifold M.

1. The mapping

$$\mathbb{R} \times M \to M, \qquad (t, p) \mapsto \theta\big(p, \exp(tX)\big)$$

is a smooth global flow for any element X of the Lie algebra $\mathrm{Lie}(G)$.

[6] [4, Theorem 20.15].

2. The mapping

$$\mathrm{Lie}(G) \to V(M), \quad X \mapsto \theta(X) = \left[p \mapsto \left(\frac{\partial}{\partial t}\right)_{t=0} \theta\big(p, \exp(tX)\big)\right] \tag{2.7}$$

is a homomorphism of Lie algebras.

DEFINITION 2.7.1
The homomorphism (2.7) is called the infinitesimal generator of the smooth right action θ.

PROPOSITION 2.7.6
Suppose that X is a smooth manifold and let V be a finite dimensional subspace of the real vector space $V(X)$. The mapping

$$V \times X \to TX, \qquad (v, x) \mapsto v(x)$$

is smooth.

Proof. Suppose that $\{e_j\}_{j=1}^n$ is a basis of the vector space V and let

$$e_j(x) = \sum_{k=1}^d e_j^k(x) \frac{\partial}{\partial x^k}.$$

We have

$$\begin{aligned} v(x) &= v^1 e_1(x) + \cdots + v^n e_n(x) \\ &= \sum_{k=1}^d \big(v^1 e_1^k(x) + \cdots + v^n e_n^k(x)\big) \frac{\partial}{\partial x^k}. \end{aligned}$$ □

COROLLARY 2.7.2
Suppose that G is a Lie group. The mapping

$$\mathrm{Lie}(G) \times G \to TG, \qquad (X, g) \mapsto X(g)$$

is smooth.

PROPOSITION 2.7.7
Suppose that X is a smooth manifold and let V be a finite dimensional subspace of the real vector space $V(X)$ such that each element of the set V is complete. The mapping

$$\mathbb{R} \times V \times X \to X, \qquad (t, v, x) \mapsto \int^t v(x)$$

is smooth.

Proof. The mapping

$$V \times X \to T_{(v,x)}(V \times X) = V \oplus T_x X, \qquad (v,x) \mapsto 0 \oplus v(x) \tag{2.8}$$

is a smooth vector field. The mapping

$$\mathbb{R} \times V \times X \to V \times X, \qquad (t,v,x) \mapsto \theta(t,v,x) = \big(v, \int^t v(x)\big)$$

is the global flow generated by the vector field (2.8) since we have

$$\theta(0,v,x) = (v,x), \qquad \begin{aligned} \frac{\partial \theta(t,v,x)}{\partial t} &= 0 \oplus \frac{\partial}{\partial t} \int^t v(x) \\ &= 0 \oplus v\big(\int^t v(x)\big). \end{aligned}$$

□

Corollary 2.7.3
The exponential mapping of a Lie group is smooth.

Proposition 2.7.8
We have $D\exp(0) = 1$ for any Lie group.

Proof. We have

$$\exp_* X = \left(\frac{\partial}{\partial t} \exp(tX)\right)_{t=0} = X$$

for any X. □

Proposition 2.7.9
The image of a smooth curve of a nonempty open interval of the real line into a smooth manifold is contained in the unique leaf of the foliation induced by an involutive distribution provided that its each velocity belongs to the involutive distribution.

Proposition 2.7.10
Suppose that G is a connected Lie group and let V be any neighborhood of the origin of the Lie algebra $\mathrm{Lie}(G)$. We have

$$G = \bigcup_{n=1}^{\infty} \{\, (\exp X_1) \cdots (\exp X_n) : (X_k)_{k=1}^n \in V^n \,\}.$$

Proposition 2.7.11
Suppose that G is a connected Lie group and let M be a smooth manifold. The mapping of the set

$$\{\,\text{smooth right actions of the Lie group } G \text{ on the manifold } M\,\}$$

into the set

$$\{\,\text{complete actions of the Lie algebra } \mathrm{Lie}(G) \text{ on the manifold } M\,\}$$

is injective.

Proof. Suppose that θ_1 and θ_2 are smooth right actions of the Lie group G on the manifold M such that we have $\theta_1(X) = \theta_2(X)$ for any element X of the Lie algebra $\mathrm{Lie}(G)$. We have

$$(\theta_1)_{\exp X} = \int^1 \theta_1(X) = \int^1 \theta_2(X) = (\theta_2)_{\exp X}$$

for any element X of the Lie algebra $\mathrm{Lie}(G)$. Suppose that g is an arbitrary element of the Lie group G. We can write $g = (\exp X_1)\cdots(\exp X_n)$ since the Lie group G is connected. We have

$$\begin{aligned}(\theta_1)_g &= (\theta_1)_{\exp X_n} \circ \cdots \circ (\theta_1)_{\exp X_1} \\ &= (\theta_2)_{\exp X_n} \circ \cdots \circ (\theta_2)_{\exp X_1} = (\theta_2)_g.\end{aligned}$$ □

Theorem 2.7.8 (Fundamental Theorem on Lie Algebra Actions)
[7] Suppose that G is a simply connected Lie group. We have

$$\begin{aligned}&\big\{\, \text{smooth right actions of the Lie group } G \text{ on a smooth manifold} \,\big\} \\ &\quad = \big\{\, \text{complete actions of the Lie algebra } \mathrm{Lie}(G) \text{ on a smooth manifold} \,\big\}.\end{aligned}$$

Proof. Suppose that θ is a complete action of the Lie algebra $\mathrm{Lie}(G)$ on a nonempty smooth manifold M. The mapping

$$\mathrm{Lie}(G) \to V(G \times M), \qquad X \mapsto X \oplus \theta(X)$$

is an injective homomorphism of Lie algebras. The set

$$\Delta = \bigsqcup_{(g,p)\in G\times M} \Delta(g,p), \qquad \Delta(g,p) = \big\{\, X \oplus \theta(X)(g,p) : X \in \mathrm{Lie}(G) \,\big\}$$

is an involutive distribution of rank $\dim G$. We write

$$\Big\{\, G_{(g,p)} : (g,p) \in G \times M \,\Big\}$$

for the foliation induced by the involutive distribution.

The mapping

$$\mathrm{Lie}(G) \times G \times M \to G \times M, \qquad (X,g,p) \mapsto \Big(g \exp X, \int^1 \theta(X)(p)\Big)$$

is smooth. The smooth mapping

$$\mathbb{R} \times G \times M \to G \times M, \qquad (t,g,p) \mapsto \Big(g \exp(tX), \int^1 \theta(tX)(p)\Big)$$

[7] [4, Theorem 20.16].

is the global flow generated by the smooth vector field $X \oplus \theta(X)$ for any element X of the Lie algebra $\mathrm{Lie}(G)$ since we have

$$\begin{aligned}\frac{\partial}{\partial t}\big(g\exp(tX), \int^1 \theta(tX)(p)\big) &= X\big(g\exp(tX)\big) \oplus \frac{\partial}{\partial t}\int^t \theta(X)(p) \\ &= X\big(g\exp(tX)\big) \oplus \theta(X)\big(\int^t \theta(X)(p)\big) \\ &= X \oplus \theta(X)\big(g\exp(tX), \int^1 \theta(tX)(p)\big).\end{aligned}$$

The point

$$\big(g\exp(tX), \int^1 \theta(tX)(p)\big)$$

belongs to the leaf $G_{(g,p)}$.

Suppose that p_0 is an arbitrary point of the manifold M. We write $G_{p_0} = G_{(e,p_0)}$ and let π_{p_0} be the ristriction of the projection π_G to the leaf G_{p_0}. The smooth mapping π_{p_0} is a submersion since we have

$$D\pi_{p_0}(g,p)\big(X \oplus \theta(X)(g,p)\big) = X(g)$$

for any element X of the Lie algebra $\mathrm{Lie}(G)$.

We show that the smooth submersion π_{p_0} is onto. Suppose that g is an arbitrary element of the Lie group G. We can write $g = (\exp X_1)\cdots(\exp X_n)$ since the Lie group G is connected. The point

$$(g,p) = \big((\exp X_1)\cdots(\exp X_n), \int^1 \theta(X_n) \circ \cdots \circ \int^1 \theta(X_1)(p_0)\big)$$

belongs to the leaf G_{p_0}.

There exists a connected neighborhood V of the origin of the Lie algebra $\mathrm{Lie}(G)$ such that the exponential mapping is a diffeomorphism of the domain V onto the domain $\exp V$. Suppose that p is an arbitrary point of the set $\pi_M\big(\pi_{p_0}^{-1}(g)\big)$. The smooth mapping

$$g\exp V \to G_{p_0}, \qquad g\exp X \mapsto \sigma_p(g\exp X) = \big(g\exp X, \int^1 \theta(X)(p)\big)$$

is an imbedding. Suppose that $(g\exp X, q)$ is an arbitrary point of the open set $\pi_{p_0}^{-1}(g\exp V)$. The point

$$(g,p) = \big(g, \int^{-1} \theta(X)(q)\big)$$

belongs to the leaf G_{p_0} and we have

$$\sigma_p(g\exp X) = \big(g\exp X, \int^1 \theta(X)(p)\big) = (g\exp X, q).$$

The mapping π_{p_0} is a smooth covering since we have

$$\pi_{p_0}^{-1}(g \exp V) = \bigsqcup_{p \in \pi_M(\pi_{p_0}^{-1}(g))} \sigma_p(g \exp V).$$

The smooth covering π_{p_0} is a diffeomorphism since the Lie group is simply connected.

We define

$$M \times G \to M, \qquad (p, g) \mapsto \theta(p, g) = \pi_M \circ \pi_p^{-1}(g). \tag{2.9}$$

We have

$$\theta\big(p, (\exp X_1) \cdots (\exp X_n)\big) = \int^1 \theta(X_n) \circ \cdots \circ \int^1 \theta(X_1)(p) \tag{2.10}$$

for any element $\big(p, (X_k)_{k=1}^n\big)$ of the set $M \times \mathrm{Lie}(G)^n$ for any n. The mapping (2.9) is a smooth right action and we have

$$\left(\frac{\partial}{\partial t}\right)_{t=0} \theta\big(p, \exp(tX)\big) = \left(\frac{\partial}{\partial t}\right)_{t=0} \int^t \theta(X)(p) = \theta(X)(p). \quad \square$$

Theorem 2.7.9

[8] Suppose that G is a simply connected Lie group. We have

$$\begin{aligned} &\Big\{ \text{homomorphisms of the Lie algebra } \mathrm{Lie}(G) \text{ into the Lie algebra } \mathrm{Lie}(H) \Big\} \\ &\qquad = \Big\{ \text{homomorphisms of the Lie group } G \text{ into the Lie group } H \Big\} \end{aligned}$$

for any Lie group H.

Proof. A homomorphism f_* of the Lie algebra $\mathrm{Lie}(G)$ into the Lie algebra $\mathrm{Lie}(H)$ is a complete action of the Lie algebra $\mathrm{Lie}(G)$ on the Lie group H. We define a smooth mapping $f(g) = f_*(e, g)$. We have

$$\begin{aligned} f_*(X) &= \left(\frac{\partial}{\partial t}\right)_{t=0} f_*\big(e, \exp(tX)\big) \\ &= \left(\frac{\partial}{\partial t}\right)_{t=0} f\big(\exp(tX)\big) \end{aligned}$$

for any element X of the Lie algebra $\mathrm{Lie}(G)$. By Proposition 2.7.5 it is sufficient to show that the smooth mapping f is a homomorphism of groups. By the equation (2.10) we have

$$\begin{aligned} f\big((\exp X_1) \cdots (\exp X_n)\big) &= \int^1 f_*(X_n) \circ \cdots \circ \int^1 f_*(X_1)(e) \\ &= \int^1 f_*(X_1)(e) \cdots \int^1 f_*(X_n)(e) \\ &= f(\exp X_1) \cdots f(\exp X_n) \end{aligned}$$

for any element $(X_k)_{k=1}^n$ of the set $\mathrm{Lie}(G)^n$ for any n. $\square$

[8] [4, Theorem 20.19].

Proposition 2.7.12
The mapping

$$G \to \big(\hom \operatorname{Lie}(G)\big)^{\times}, \qquad g \mapsto \operatorname{Ad} g = \left[X \mapsto \left(\frac{\partial}{\partial t}\right)_{t=0} g(\exp tX)g^{-1}\right]$$

is a representation of a Lie group G.

Theorem 2.7.10
We have $D(\operatorname{Ad})(e) = \operatorname{ad}$ for any Lie group.

Chapter 3

Root Systems and Semisimple Lie Algebras

3.1 Zariski Topology

Suppose that V is a finite dimensional vector space over a field $\mathbb{F}$ and let $(e^k)_{k=1}^n$ be the dual basis of some basis $(e_k)_{k=1}^n$ of the vector space.

Proposition 3.1.1
The mapping

$$S(V^*) \to \mathbb{F}, \qquad f \mapsto f(x)$$

is a homomorphism of algebras with identity for any point x of the space V.

Proposition 3.1.2
There exists a canonical homomorphism of algebras with identity of the algebra $S(V^*)$ into the algebra $\mathbb{F}^V$.

Definition 3.1.1
Suppose that S is a subset of the algebra $S(V^*)$. We write

$$S^{-1}(0) = \bigcap_{f \in S} f^{-1}(0).$$

Proposition 3.1.3
Suppose that S is a subset of the algebra $S(V^*)$. We have

$$S^{-1}(0) = (S)^{-1}(0).$$

Proposition 3.1.4
Suppose that $(S_k)_{k=1}^m$is a finite sequence of subsets of the algebra $S(V^*)$. We have

$$\bigcup_{k=1}^m S_k^{-1}(0) = (S_1 \cdots S_m)^{-1}(0).$$

Proposition 3.1.5
The collection

$$\{\, V \setminus S^{-1}(0) : S \text{ is a subset of the algebra } S(V^*) \,\}$$

is a topology of the space V.

Proposition 3.1.6
Suppose that the field $\mathbb{F}$ is infinite and let f be an element of the space $S(V^*) \setminus \{0\}$.

1. The Zariski open set $V \setminus f^{-1}(0)$ is not empty.
2. The Zariski open set $V \setminus f^{-1}(0)$ is infinite if we have $\dim V \geq 1$.

Proof. The proof is by induction on the nonnegative integer $n = \dim V$. Suppose that we have $n = 0$. The Zariski closed set $f^{-1}(0)$ is empty. Suppose that we have $n > 0$. We can write

$$f(e) = f_m(e^1, \dots, e^{n-1})(e^n)^m + \cdots + f_0(e^1, \dots, e^{n-1}), \quad f_m(e^1, \dots, e^{n-1}) \neq 0.$$

There exists an element $(x^1, \dots, x^{n-1})$ of the space $\mathbb{F}^{n-1}$ such that we have $f_m(x^1, \dots, x^{n-1}) \neq 0$ by the induction hypothesis. We define

$$g(e) = f(x^1, \dots, x^{n-1}, e) = f_m(x^1, \dots, x^{n-1})e^m + \cdots + f_0(x^1, \dots, x^{n-1}).$$

The Zariski open set $\mathbb{F} \setminus g^{-1}(0)$ is infinite. □

Proposition 3.1.7
Suppose that the field $\mathbb{F}$ is infinite. Any finite intersection of nonempty Zariski open sets of the space V is not empty.

Proposition 3.1.8
Suppose that the field $\mathbb{F}$ is infinite. The algebra $S(V^*)$ is a subalgebra with identity of the algebra $\mathbb{F}^V$.

Proposition 3.1.9
We have

$$\big((e^1 - x^1, \dots, e^n - x^n)\big)^{-1}(0) = \{x\}$$

for any point $x = \sum_{k=1}^n x^k e_k$ of the space V.

Definition 3.1.2
We define a linear mapping

$$\frac{\partial}{\partial x_j}$$

on the algebra $\mathbb{F}[x_1, \dots, x_n]$ by

$$\frac{\partial x_1^{m_1} \cdots x_n^{m_n}}{\partial x_j} = \frac{m_j x_1^{m_1} \cdots x_n^{m_n}}{x_j}$$

for any m.

Proposition 3.1.10
The linear mapping $\dfrac{\partial}{\partial x_j}$ is a derivation of the algebra $\mathbb{F}[x_1, \ldots, x_n]$.

Proof. We show that we have

$$\frac{\partial f(x)g(x)}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} g(x) + f(x) \frac{\partial g(x)}{\partial x_j}$$

for any elements $f(x)$ and $g(x)$ of the algebra $\mathbb{F}[x_1, \ldots, x_n]$. We may assume that the polynomials $f(x)$ and $g(x)$ are monomials since the mappings

$$\big(f(x), g(x)\big) \mapsto \frac{\partial f(x)g(x)}{\partial x_j}, \qquad \big(f(x), g(x)\big) \mapsto \frac{\partial f(x)}{\partial x_j} g(x) + f(x) \frac{\partial g(x)}{\partial x_j}$$

are bilinear. □

Proposition 3.1.11
The vector space V is a subspace of the vector space $\operatorname{Der} S(V^*)$.

Suppose that W is a finite dimensional vector space over the field $\mathbb{F}$ and let $(e^k)_{k=1}^m$ be the dual basis of some basis $(e_k)_{k=1}^m$ of the vector space.

Proposition 3.1.12
There exists a canonical linear mapping of the space $S(V^*) \otimes W$ into the space $W^V = \mathbb{F}^V \otimes W$.

Proposition 3.1.13
Suppose that the field $\mathbb{F}$ is infinite. The space $S(V^*) \otimes W$ is a subspace of the space W^V.

Proposition 3.1.14
Suppose that X is a commutative algebra with identity over the field $\mathbb{F}$.

1. There exists a canonical mapping

$$S(V^*) \times (X \otimes V) \to X,$$
$$(f, x) = \Big(f(e^1, \ldots, e^n), \sum_{k=1}^{n} x^k \otimes e_k\Big) \mapsto f(x) = f(x^1, \ldots, x^n).$$

2. The mapping

$$S(V^*) \to X, \qquad f \mapsto f(x)$$

is a homomorphism of algebras with identity for any element x of the space $X \otimes V$.

Definition 3.1.3
We write

$$\begin{aligned} g \circ f &= g(f) \\ &= g\big(f^1(e^1, \dots, e^n), \dots, f^m(e^1, \dots, e^n)\big) \end{aligned}$$

for any element

$$(g, f) = \Big(g(e^1, \dots, e^m), \sum_{k=1}^{m} f^k(e^1, \dots, e^n) \otimes e_k\Big)$$

of the space $S(W^*) \times \big(S(V^*) \otimes W\big)$.

Suppose that U is a finite dimensional vector space over the field $\mathbb{F}$.

Proposition 3.1.15
There exists a canonical mapping

$$S(V^*) \otimes W \times S(U^*) \otimes V \to S(U^*) \otimes W,$$
$$(g = \sum_{k=1}^{m} g^k \otimes e_k, f) \mapsto g \circ f = \sum_{k=1}^{m} (g^k \circ f) \otimes e_k.$$

Proposition 3.1.16
The following diagram commutes.

$$\begin{array}{ccc} S(V^*) \otimes W \times S(U^*) \otimes V & \longrightarrow & S(U^*) \otimes W \\ \downarrow & & \downarrow \\ W^V \times V^U & \longrightarrow & W^U \end{array}$$

Proposition 3.1.17
We have the following.

1. There exists a canonical mapping

$$S(V^*) \otimes W \to S(V^*) \otimes \hom(V, W), \qquad f \mapsto f'.$$

2. We have

$$f' = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial f^i(e^1, \dots, e^n)}{\partial e^j} e_i \otimes e^j$$

for any element

$$f = \sum_{k=1}^{m} f^k(e^1, \dots, e^n) \otimes e_k$$

of the space $S(V^*) \otimes W$.

3. There exists a canonical mapping
$$S(V^*)\otimes V \to S(V^*), \qquad f\mapsto \det f'.$$

4. We have
$$\det f' = \det\left(\frac{\partial f^i(e^1,\dots,e^n)}{\partial e^j}\right)_{i,j=1}^n$$
for any element
$$f=\sum_{k=1}^n f^k(e^1,\dots,e^n)\otimes e_k$$
of the space $S(V^*)\otimes V$.

Proposition 3.1.18
Suppose that X is an algebra over the field $\mathbb{F}$. There exists a canonical linear mapping
$$\begin{aligned}X\otimes \hom(V,W)\otimes X\otimes \hom(U,V)&\to X\otimes\hom(U,W),\\ x_1\otimes g\otimes x_2\otimes f&\mapsto (x_1\otimes g)\circ(x_2\otimes f)=x_1x_2\otimes(g\circ f).\end{aligned}$$

Theorem 3.1.1
Suppose that (g,f) is an element of the space $S(V^*)\otimes W\times S(U^*)\otimes V$. We have
$$(g\circ f)'=(g'\circ f)\circ f'.$$

Proposition 3.1.19
Suppose that f is an element of the space $S(V^*)\otimes V$. We have
$$(\det f')(x)=\det\bigl(f'(x)\bigr)$$
for any point x of the space V.

Proposition 3.1.20
Suppose that the field $\mathbb{F}$ is of characteristic 0. An element f of the space $S(V^*)\otimes V$ is algebraically independent over the field $\mathbb{F}$ if we have $\det f'\neq 0$.

Proof. Suppose that the element f is algebraically dependent over the field $\mathbb{F}$. We show that we have $\det f'=0$. There exists an element g of the space $S(V^*)\setminus\{0\}$ such that we have $g\circ f=0$. We may assume that we have
$$\begin{aligned}\deg g=\min\bigl\{\deg g: g \text{ is an element of the set } S(V^*)\setminus\{0\}\\ \text{such that we have } g\circ f=0\bigr\}\geq 1.\end{aligned}$$
We have
$$V=\bigl\{\,x:(g'\circ f)(x)=0\,\bigr\}\cup\bigl\{\,x:\det f'(x)=0\,\bigr\}$$
since we have
$$0=(g\circ f)'=(g'\circ f)\circ f'.$$

Suppose that we have $g' \circ f = 0$. We have $\dfrac{\partial g}{\partial e^1} = \cdots = \dfrac{\partial g}{\partial e^n} = 0$ since we have

$$\deg \frac{\partial g}{\partial e^1}, \ldots, \deg \frac{\partial g}{\partial e^n} < \deg g.$$

The element g is constant since the field $\mathbb{F}$ is of characteristic 0. This is a contradiction. We have $\det f' = 0$. □

3.2 Commutative Algebras

In this section we require the following.

1. An algebra is a commutative algebra with identity.
2. A subalgebra is a subalgebra with identity.
3. A homomorphism of algebras is a homomorphism of algebras with identity.

Suppose that R_1 is a subring of a ring R_2.

Definition 3.2.1
An element x of the ring R_2 is said to be integral over the ring R_1 if there exists a monic polynomial f over the ring R_1 such that we have $f(x) = 0$.

Proposition 3.2.1
Suppose that S is a subset of the ring R_2 that is integral over the ring R_1. The ring $R_1[S]$ is integral over the ring R_1.

Proposition 3.2.2
Suppose that $\mathbb{E}$ is an extension field of a field $\mathbb{F}$. An element of the field $\mathbb{E}$ is algebraic over the field $\mathbb{F}$ if and only if it is integral over the ring $\mathbb{F}$.

Proposition 3.2.3
Suppose that $\mathbb{E}$ is an extension field of a field $\mathbb{F}$ and let S be a subset of the field $\mathbb{E}$ that is algebraic over the field $\mathbb{F}$. The field $\mathbb{F}(S) = \mathbb{F}[S]$ is algebraic over the field $\mathbb{F}$.

Proposition 3.2.4
Suppose that R is a subring of a field $\mathbb{F}$. The field $Q(R)$ is a subfield of the field $\mathbb{F}$.

Proposition 3.2.5
Suppose that the ring R_2 is an integral domain. The field $Q(R_1)$ is a subfield of the field $Q(R_2)$ and the following diagram commutes.

$$\begin{array}{ccc} R_1 & \longrightarrow & R_2 \\ \downarrow & & \downarrow \\ Q(R_1) & \longrightarrow & Q(R_2) \end{array}$$

PROPOSITION 3.2.6
Suppose that the ring R_2 is an integral domain that is integral over the integral domain R_1. The field $Q(R_2)$ is algebraic over the field $Q(R_1)$.

Proof. The field $Q(R_2) = Q(R_1)(R_2)$ is algebraic over the field $Q(R_1)$ since the integral domain R_2 is algebraic over the field $Q(R_1)$. □

PROPOSITION 3.2.7
Suppose that the ring R_2 is an integral domain that is integral over the integral domain R_1. The integral domain R_1 is a field if and only if the integral domain R_2 is a field.

Proof. Suppose that the integral domain R_1 is a field. The integral domain R_2 is algebraic over the field R_1 since the field $Q(R_2)$ is algebraic over the field R_1. We have $Q(R_2) = R_1(R_2) = R_1[R_2] = R_2$.

Suppose that the integral domain R_2 is a field and let x be an arbitrary element of the set $R_1 \setminus \{0\}$. We can write

$$(x^{-1})^n + r_1(x^{-1})^{n-1} + \cdots + r_n = 0$$

since the element x^{-1} is integral over the integral domain R_1. The element

$$x^{-1} = -(r_1 + \cdots + r_n x^{n-1})$$

belongs to the integral domain R_1. □

THEOREM 3.2.1
Suppose that R is an integral domain and let S be a finite set that is algebraic over the field $Q(R)$. There exists an element x of the set $R \setminus \{0\}$ such that the integral domain $R[x^{-1}][S]$ is integral over the integral domain $R[x^{-1}]$.

PROPOSITION 3.2.8
An ideal P of a ring R is prime if and only if the ring R/P is an integral domain.

PROPOSITION 3.2.9
An ideal M of a ring R is maximal if and only if the ring R/M is a field.

PROPOSITION 3.2.10
Any algebraically closed field is infinite.

Proof. It is sufficient to show that the algebraic closure of the field $\mathbb{F}_p$ is infinite for any prime number p. The algebraic closure of the field $\mathbb{F}_p$ is the algebraic closure of the field $\mathbb{F}_{p^n}$ for any positive integer n. □

PROPOSITION 3.2.11
Suppose that $\mathbb{F}$ is a field. We have

$$\mathbb{F}[x][\frac{1}{f(x)}] \neq \mathbb{F}(x)$$

for any element $f(x)$ of the set $\mathbb{F}[x] \setminus \{0\}$.

Proof. We have

$$\mathbb{F}[x][\frac{1}{f(x)}] = \lim_{n\to\infty} \frac{\mathbb{F}[x]}{f(x)^n}. \tag{3.1}$$

There exists an element x_0 of the algebraic closure $\overline{\mathbb{F}}$ such that we have $f(x_0) \neq 0$ since the algebraic closure $\overline{\mathbb{F}}$ is infinite by Proposition 3.2.10. We write $f_0(x)$ for the minimal polynomial of the element x_0 over the field $\mathbb{F}$. Suppose that the element $f_0(x)^{-1}$ is contained in the integral domain (3.1). There exists an element $g(x)$ of the integral domain $\mathbb{F}[x]$ such that we have

$$\frac{1}{f_0(x)} = \frac{g(x)}{f(x)^n}.$$

We have $f(x_0)^n = f_0(x_0)g(x_0) = 0$. This is a contradiction. □

Theorem 3.2.2

Suppose that $\mathbb{E}$ is an extension field of a field $\mathbb{F}$. The field $\mathbb{E}$ is finite dimensional over the field $\mathbb{F}$ if the ring $\mathbb{E}$ is finitely generated over the field $\mathbb{F}$.

Proof. We can write $\mathbb{E} = \mathbb{F}[x_1, \dots, x_n]$ and the proof is by induction on the nonnegative integer n. Suppose that we have $n > 0$. The field $\mathbb{E}$ is finite dimensional over the field $\mathbb{F}(x_1)$ by the induction hypothesis. It is sufficient to show that the element x_1 is algebraic over the field $\mathbb{F}$. Suppose contrary. There exists an element $f(x_1)$ of the set $\mathbb{F}[x_1] \setminus \{0\}$ such that the field $\mathbb{E}$ is integral over the integral domain

$$\mathbb{F}[x_1][\frac{1}{f(x_1)}] \neq \mathbb{F}(x_1) \tag{3.2}$$

by Theorem 3.2.1. The integral domain (3.2) is a field by Proposition 3.2.7. This is a contradiction. □

Proposition 3.2.12

Suppose that V is a finite dimensional vector space over an algebraically closed $\mathbb{F}$ and let M be a maximal ideal of the algebra $S(V^*)$. We have $S(V^*)/M = \mathbb{F}$.

Proof. We can write $S(V^*) = \mathbb{F}[x_1, \dots, x_n]$. We have

$$\frac{\mathbb{F}[x_1, \dots, x_n]}{M} = \mathbb{F}[x_1 + M, \dots, x_n + M].$$

The field $\mathbb{F}[x_1+M, \dots, x_n+M]$ is finite dimensional over the field $\mathbb{F}$ by Theorem 3.2.2. We have $\mathbb{F}[x_1 + M, \dots, x_n + M] = \mathbb{F}$ since the field $\mathbb{F}$ is algebraically closed. □

Proposition 3.2.13

Suppose that $\mathbb{F}$ is a field and let ν be a point of the space $\mathbb{F}^n$.

1. We have

$$\begin{aligned}(x_1 - \nu_1, \dots, x_n - \nu_n) &= \mathbb{F}[x_1](x_1 - \nu_1) + \cdots + \mathbb{F}[x_1, \dots, x_n](x_n - \nu_n) \\ &= \{\, f(x) \in \mathbb{F}[x_1, \dots, x_n] : f(\nu) = 0 \,\}. \end{aligned} \tag{3.3}$$

2. The ideal (3.3) is maximal.

Proof. The ideal (3.3) is maximal since we have

$$\frac{\mathbb{F}[x_1, \ldots, x_n]}{(x_1 - \nu_1, \ldots, x_n - \nu_n)} = \mathbb{F}. \quad \square$$

Proposition 3.2.14
Any finite dimensional vector space is contained in the set of maximal ideals of the symmetric algebra of the dual space.

Theorem 3.2.3
Any finite dimensional vector space over an algebraically closed field is the set of maximal ideals of the symmetric algebra of the dual space.

Proposition 3.2.15
Suppose that the ring R_2 is integral over the ring R_1 and let S be a proper multiplicative subset of the ring R_1. The ring R_2S^{-1} is an integral extension of the ring R_1S^{-1}.

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d] & R_2 \arrow[d] \\ R_1S^{-1} \arrow[r] & R_2S^{-1} \end{tikzcd}$$

Proposition 3.2.16
Suppose that the ring R_2 is integral over the ring R_1 and let J be an ideal of the ring R_2. The ring R_2/J is an integral extension of the ring $R_1/R_1 \cap J$.

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d] & R_2 \arrow[d] \\ R_1/R_1 \cap J \arrow[r] & R_2/J \end{tikzcd}$$

Proposition 3.2.17
Suppose that the ring R_2 is integral over the ring R_1 and let P be a prime ideal of the ring R_2. The ideal P is maximal if and only if the ideal $R_1 \cap P$ is maximal.

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d] & R_2 \arrow[d] \\ R_1/R_1 \cap P \arrow[r] & R_2/P \end{tikzcd}$$

Definition 3.2.2
A ring is said to be local if it has a unique maximal ideal.

Proposition 3.2.18
Suppose that P is a prime ideal of a ring R.

1. The set $R \setminus P$ is a proper multiplicative subset of the ring R.

2. The ring $R(R \setminus P)^{-1}$ is local and let M be its unique maximal ideal. We have
$$Q(R/P) = R(R \setminus P)^{-1}/M.$$

Proposition 3.2.19
Any proper ideal of a ring is contained in a maximal ideal.

Corollary 3.2.1
Any nontrivial ring has a maximal ideal.

Theorem 3.2.4
Suppose that the ring R_2 is integral over the ring R_1 and let P_1 be a prime ideal of the ring R_1. There exists a prime ideal P_2 of the ring R_2 such that we have $P_1 = R_1 \cap P_2$.

Proof.
$$\begin{tikzcd} R_1 \arrow[r] \arrow[d, "f_1"'] & R_2 \arrow[d, "f_2"] \\ R_1(R_1 \setminus P_1)^{-1} \arrow[r] & R_2(R_1 \setminus P_1)^{-1} \end{tikzcd}$$
The ring $R_2(R_1 \setminus P_1)^{-1}$ has a maximal ideal M_2 and let
$$M_1 = R_1(R_1 \setminus P_1)^{-1} \cap M_2$$
be the unique maximal ideal of the local ring $R_1(R_1 \setminus P_1)^{-1}$.
$$\begin{tikzcd} R_1 \arrow[r] \arrow[d, "f_1"'] & R_2 \arrow[d, "f_2"] \\ R_1(R_1 \setminus P_1)^{-1} \arrow[r] \arrow[d] & R_2(R_1 \setminus P_1)^{-1} \arrow[d] \\ R_1(R_1 \setminus P_1)^{-1}/M_1 \arrow[r] & R_2(R_1 \setminus P_1)^{-1}/M_2 \end{tikzcd}$$
The ideal $P_2 = f_2^{-1}(M_2)$ is prime and we have $P_1 = f_1^{-1}(M_1) = R_1 \cap P_2$. $\square$

$$\begin{tikzcd} R_1/P_1 \arrow[r] \arrow[d] & R_2/P_2 \arrow[d] \\ R_1(R_1 \setminus P_1)^{-1}/M_1 \arrow[r] & R_2(R_1 \setminus P_1)^{-1}/M_2 \end{tikzcd}$$

Theorem 3.2.5
A homomorphism of a ring into a field extends to a homomorphism of any integral extension into the algebraic closure.

Proof. Suppose that the ring R_2 is integral over the ring R_1 and let f be a homomorphism of the ring R_1 into a field $\mathbb{F}$.

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d, "f"'] & R_2 \\ \overline{\mathbb{F}} & \end{tikzcd}$$

The ideal $P_1 = \ker f$ is prime and there exists a prime ideal P_2 of the ring R_2 such that we have $P_1 = R_1 \cap P_2$. The field $Q(R_2/P_2)$ is algebraic over the field $Q(R_1/P_1)$. The algebraic closure of the field $Q(R_1/P_1)$ is the algebraic closure of the field $Q(R_2/P_2)$.

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d] & R_2 \arrow[d] & & \\ R_1/P_1 \arrow[r] \arrow[d] & R_2/P_2 \arrow[d] & & \\ Q(R_1/P_1) \arrow[r] & Q(R_2/P_2) \arrow[r] & \overline{Q(R_1/P_1)} \arrow[r] & \overline{\mathbb{F}} \end{tikzcd}$$

□

PROPOSITION 3.2.20

Suppose that S is a proper multiplicative subset of an integral domain R. The integral domain RS^{-1} is a subring of the field $Q(R)$ and we have $RS^{-1} = R[S^{-1}]$.

PROPOSITION 3.2.21

Suppose that A is an algebra over a field and let S be a proper multiplicative subset. The ring AS^{-1} is an algebra and the canonical mapping of the algebra A into the algebra AS^{-1} is a homomorphism of algebras.

THEOREM 3.2.6

Suppose that $\mathbb{F}$ is an algebraically closed field and let

$$f(x) = \big(f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)\big)$$

be an element of the space $\mathbb{F}[x_1, \dots, x_n]^n$ that is algebraically independent over the field $\mathbb{F}$. The set $f(\mathbb{F}^n)$ has nonempty Zariski interior.

Proof. The element $f(x)$ is a transcendence basis of the field $\mathbb{F}(x)$ over the field $\mathbb{F}$ since the transcendence degree of the field $\mathbb{F}(x)$ over the field $\mathbb{F}$ is n. There exists an element $g(x)$ of the set $\mathbb{F}[x] \setminus \{0\}$ such that the integral domain

$$\mathbb{F}\big[x, \frac{1}{g \circ f(x)}\big]$$

is integral over the integral domain

$$\mathbb{F}\big[f(x), \frac{1}{g \circ f(x)}\big]$$

by Theorem 3.2.1 since the field $\mathbb{F}(x)$ is algebraic over the field $\mathbb{F}\big(f(x)\big)$. We show that the set $f(\mathbb{F}^n)$ contains the nonempty Zariski open set $\mathbb{F}^n \setminus g^{-1}(0)$. Suppose that ν is an arbitrary element of the set $\mathbb{F}^n \setminus g^{-1}(0)$. There exists a unique homomorphism of algebras

$$\mathbb{F}\big[f(x)\big] \to \mathbb{F}, \qquad f(x) \mapsto \nu.$$

The homomorphism extends uniquely to a homomorphism of algebras

$$\mathbb{F}\Big[f(x), \frac{1}{g \circ f(x)}\Big] \to \mathbb{F}, \qquad \frac{1}{g \circ f(x)} \mapsto \frac{1}{g(\nu)}.$$

The homomorphism extends to a homomorphism of algebras

$$\mathbb{F}\Big[x, \frac{1}{g \circ f(x)}\Big] \to \mathbb{F}, \qquad x \mapsto \mu.$$

by Theorem 3.2.5 since the field $\mathbb{F}$ is algebraically closed. We have $\nu = f(\mu)$. $\square$

Corollary 3.2.2
Suppose that V is a finite dimensional vector space over an algebraically closed field of characteristic 0 and let f be an element of the space $S(V^*) \otimes V$ such that we have $\det f' \neq 0$. The set $f(V)$ has nonempty Zariski interior.

Proof. By Proposition 3.1.20. $\square$

Proposition 3.2.22
Any vector space over an infinite field is not a finite union of proper subspaces.

3.3 Cartan Subalgebras

Proposition 3.3.1
Suppose that f is a homomorphism of a Lie algebra into an algebra. We have

$$\begin{aligned}&\big(f(x) - (\mu+\nu)\big) f\big((\operatorname{ad} x - \mu)^n y\big) \\ &\qquad = f\big((\operatorname{ad} x - \mu)^{n+1} y\big) + f\big((\operatorname{ad} x - \mu)^n y\big)\big(f(x) - \nu\big)\end{aligned}$$

for any elements x and y of the Lie algebra and for any scalars μ and ν for any nonnegative integer n.

Proof. We have

$$\begin{aligned}&\big(f(x) - (\mu+\nu)\big) f\big((\operatorname{ad} x - \mu)^n y\big) \\ &\quad = f(x) f\big((\operatorname{ad} x - \mu)^n y\big) - (\mu+\nu) f\big((\operatorname{ad} x - \mu)^n y\big) \\ &\quad = f\big((\operatorname{ad} x - \mu)^{n+1} y\big) + f\big((\operatorname{ad} x - \mu)^n y\big)\big(f(x) - \nu\big).\end{aligned}$$

$\square$

Proposition 3.3.2
Suppose that f is a homomorphism of a Lie algebra into an algebra. We have

$$\big(f(x)-(\mu+\nu)\big)^n f(y)=\sum_{k=0}^{n}\binom{n}{k} f\big((\operatorname{ad} x-\mu)^k y\big)\big(f(x)-\nu\big)^{n-k}$$

for any elements x and y of the Lie algebra and for any scalars μ and ν for any nonnegative integer n.

Proof. The proof is by induction on the nonnegative integer n. Suppose that we have $n>0$. We have

$$\begin{aligned}
&\big(f(x)-(\mu+\nu)\big)^n f(y)\\
&\quad=\big(f(x)-(\mu+\nu)\big)\sum_{k=0}^{n-1}\binom{n-1}{k} f\big((\operatorname{ad} x-\mu)^k y\big)\big(f(x)-\nu\big)^{n-k-1}\\
&\quad=\sum_{k=0}^{n-1}\binom{n-1}{k}\Big(f\big((\operatorname{ad} x-\mu)^{k+1} y\big)\big(f(x)-\nu\big)^{n-k-1}+f\big((\operatorname{ad} x-\mu)^k y\big)\big(f(x)-\nu\big)^{n-k}\Big)\\
&\quad=\sum_{k=0}^{n}\binom{n}{k} f\big((\operatorname{ad} x-\mu)^k y\big)\big(f(x)-\nu\big)^{n-k}.
\end{aligned}$$

□

Proposition 3.3.3
Suppose that ρ is a representation of a nilpotent Lie algebra.

$$\lim_{n\to\infty}\ker\big(\rho(x)-\nu\big)^n \tag{3.4}$$

is an invariant subspace for the representation ρ for any element x of the Lie algebra and for any scalar ν.

Proof. Suppose that v is an arbitrary element of the subspace (3.4). We have

$$\big(\rho(x)-\nu\big)^n\rho(y)v=\sum_{k=0}^{n}\binom{n}{k}\rho\big((\operatorname{ad} x)^k y\big)\big(\rho(x)-\nu\big)^{n-k}v=0$$

eventually for any element y of the Lie algabra since we have $(\operatorname{ad} x)^k=0$ eventually. □

Definition 3.3.1
A one dimensional representation μ of a Lie algebra is called a weight of a representation ρ if the subspace

$$\bigcap_x \lim_{n\to\infty}\ker\big(\rho(x)-\mu(x)\big)^n\neq\{0\}.$$

Proposition 3.3.4
Suppose that x is a linear mapping on a finite dimensional vector space V over an algebraically closed field $\mathbb{F}$. We have

$$V=\bigoplus_{\nu\in\mathbb{F}}\lim_{n\to\infty}\ker(x-\nu)^n.$$

Definition 3.3.2

A representation of a Lie algebra on a nontrivial vector space V is said to be indecomposable if there do not exist nontrivial invariant subspaces V_1 and V_2 such that we have $V = V_1 \oplus V_2$.

Proposition 3.3.5

Suppose that ρ is a finite dimensional indecomposable representation of a nilpotent Lie algebra over an algebraically closed field of characteristic 0. There exists a unique weight of the representation ρ.

Proof. There exists a unique scalar $\mu(x)$ such that the invariant subspace

$$\lim_{n\to\infty} \ker\bigl(\rho(x) - \mu(x)\bigr)^n \neq \{0\}$$

for any element x of the Lie algebra. It is sufficient to show that the functional μ is a homomorphism of Lie algebras. The functional μ is a homomorphism of Lie algebras since each $\rho(x) - \mu(x)$ is strictly upper triangular by Lie's theorem. □

Theorem 3.3.1

Suppose that ρ is a representation of a nilpotent Lie algebra over an algebraically closed field of characteristic 0 on a finite dimensional vector space V. We have

$$V = \bigoplus_{\mu \text{ is a weight}} \bigcap_x \lim_{n\to\infty} \ker\bigl(\rho(x) - \mu(x)\bigr)^n.$$

Proposition 3.3.6

The normaliser of a Lie subalgebra is a Lie subalgebra.

Proposition 3.3.7

A Lie subalgebra is an ideal of its normaliser.

Definition 3.3.3

A nilpotent Lie subalgebra that contains the normaliser of the Lie subalgebra is called a Cartan subalgebra.

Definition 3.3.4

Suppose that x is an element of a Lie algebra g. We define a subspace

$$g^x_\nu = \lim_{n\to\infty} \ker(\operatorname{ad} x - \nu)^n$$

for any scalar ν.

Proposition 3.3.8

Suppose that x is an element of a Lie algebra g. The set $\bigl[g^x_\mu, g^x_\nu\bigr]$ is contained in the subspace $g^x_{\mu+\nu}$ for any scalars μ and ν.

Proof. Suppose that (y, z) is an arbitrary element of the set $g^x_\mu \times g^x_\nu$. We have

$$\bigl(\operatorname{ad} x - (\mu+\nu)\bigr)^n[y, z] = \sum_{k=0}^n \binom{n}{k} \bigl[(\operatorname{ad} x - \mu)^k y, (\operatorname{ad} x - \nu)^{n-k} z\bigr] = 0$$

eventually. □

PROPOSITION 3.3.9
An element x of a Lie algebra g belongs to the Lie subalgebra g_0^x.

PROPOSITION 3.3.10
Suppose that x is an element of a Lie algebra g. The Lie subalgebra g_0^x contains the normaliser of the Lie subalgebra g_0^x.

Proof. Suppose that y is an element of the normaliser of the Lie subalgebra g_0^x. The element $(\operatorname{ad} x)(y)$ belongs to the Lie subalgebra g_0^x. The element y belongs to the Lie subalgebra g_0^x. □

DEFINITION 3.3.5
Suppose that g is a finite dimensional Lie algebra over an algebraically closed field. We define $\operatorname{rank} g = \min_x \dim g_0^x$.

DEFINITION 3.3.6
An element x of a finite dimensional Lie algebra g over an algebraically closed field is said to be regular if we have $\dim g_0^x = \operatorname{rank} g$.

PROPOSITION 3.3.11
The set of regular elements of a finite dimensional Lie algebra over an algebraically closed field is a nonempty Zariski open set.

Proof. There exists a unique element f of the set $S(g^*) \setminus \{0\}$ such that the characteristic polynomial of the derivation $\operatorname{ad} x$ is given by

$$\det(t - \operatorname{ad} x) = t^{\dim g} + \cdots + f(x)t^{\operatorname{rank} g}$$

for any element x of the Lie algebra g. The set of regular elements is the nonempty Zariski open set $g \setminus f^{-1}(0)$. □

THEOREM 3.3.2
Suppose that g is a finite dimensional Lie algebra over an algebraically closed field and let x be its regular element. The Lie subalgebra g_0^x is a Cartan subalgebra.

Proof. Suppose that x is an arbitrary element of the Lie algebra g. The Zariski open set

$$U_0 = \left\{ x' \in g_0^x : \operatorname{ad}_{g_0^x} x' \text{ is not nilpotent} \right\}$$

is empty if and only if the Lie subalgebra g_0^x is nilpotent by Engel's theorem. The Zariski open set

$$U_0^\perp = \bigcap_{\nu \in \mathbb{F}^\times} \left\{ x' \in g_0^x : \operatorname{ad}_{g_\nu^x} x' \text{ is regular} \right\}$$

is not empty since it contains the element x. It is sufficient to show that the element x is not regular if the Lie subalgebra g_0^x is not nilpotent. The nonempty Zariski open sets U_0 and $U_0^\perp$ have a point in common by Proposition 3.1.7 and Proposition 3.2.10. We have $\operatorname{rank} g < \dim g_0^x$. □

Corollary 3.3.1
Any finite dimensional Lie algebra over an algebraically closed field has a Cartan subalgebra.

Proposition 3.3.12
Suppose that x and y are nilpotent elements of an algebra with identity over a field of characteristic 0 such that we have $[x, y] = 0$. The element $x + y$ is nilpotent and we have $\exp(x + y) = (\exp x)(\exp y)$.

Proof. There exists a nonnegative integer m such that we have $x^{m+1} = y^{m+1} = 0$ since the elements x and y are nilpotent. We have

$$\begin{aligned}\exp(x+y) &= \sum_{n=0}^{2m} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \sum_{m=0}^{2m} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} \\ &= \sum_{n=0}^{m} \frac{x^n}{n!} \sum_{n=0}^{m} \frac{y^n}{n!} = (\exp x)(\exp y)\end{aligned}$$

since we have $[x, y] = 0$. □

Proposition 3.3.13
Suppose that x is a nilpotent elements of an algebra with identity over a field of characteristic 0. The element $\exp x$ is invertible.

Proof. We have $(\exp x)^{-1} = \exp(-x)$. □

Proposition 3.3.14
Suppose that D is a nilpotent derivation of a Lie algebra over a field of characteristic 0. The element $\exp D$ is an automorphism of the Lie algebra.

Proof. It is sufficient to show that the element $\exp D$ is a homomorphism of Lie algebras. We write m for the Lie bracket. We have

$$D \circ m = m \circ (D \otimes 1 + 1 \otimes D)$$

since we have

$$\begin{aligned}D \circ m(x \otimes y) &= m(Dx \otimes y + x \otimes Dy) \\ &= m \circ (D \otimes 1 + 1 \otimes D)(x \otimes y)\end{aligned}$$

for any elements x and y of the Lie algebra. We have

$$\exp(D \otimes 1 + 1 \otimes D) = (\exp D) \otimes (\exp D)$$

since the elements $D \otimes 1$ and $1 \otimes D$ are nilpotent and commuting. We have

$$\begin{aligned}(\exp D) \circ m &= \sum_{n=0}^{\infty} \frac{D^n \circ m}{n!} \\ &= \sum_{n=0}^{\infty} \frac{m \circ (D \otimes 1 + 1 \otimes D)^n}{n!} \\ &= m \circ \exp(D \otimes 1 + 1 \otimes D) = m \circ (\exp D) \otimes (\exp D).\end{aligned}$$ □

Definition 3.3.7
We write $\operatorname{Der} g$ for the set of derivations of a Lie algebra g.

Proposition 3.3.15
Suppose that g is a Lie algebra. The set $\operatorname{Der} g$ is a Lie subalgebra of the Lie algebra $\hom g$.

Proof. Suppose that D_1 and D_2 are derivations of the Lie algebra g. We have

$$\begin{aligned}[D_1, D_2][x, y] &= (D_1 \circ D_2 - D_2 \circ D_1)[x, y] \\ &= D_1\big([D_2 x, y] + [x, D_2 y]\big) - D_2\big([D_1 x, y] + [x, D_1 y]\big) \\ &= \big[[D_1, D_2]x, y\big] + \big[x, [D_1, D_2]y\big]\end{aligned}$$

for any elements x and y of the Lie algebra g. □

Definition 3.3.8
Suppose that g is a Lie algebra. The Lie algebra $\operatorname{Der} g$ is called the derivation algebra of the Lie algebra g.

Proposition 3.3.16
The adjoint representation of a Lie algebra g is a homomorphism of Lie algebras of the Lie algebra g into the derivation algebra $\operatorname{Der} g$.

Definition 3.3.9
The group of inner automorphisms of a Lie algebra over a field of characteristic 0 is the group generated by the set

$$\big\{ \exp(\operatorname{ad} x) : x \text{ is an element of the Lie group} \\ \text{such that the derivation } \operatorname{ad} x \text{ is nilpotent} \big\}.$$

Definition 3.3.10
We write $\operatorname{Inn} g$ for the group of inner automorphisms of a Lie algebra g.

Proposition 3.3.17
A proper Lie subalgebra of a nilpotent Lie algebra is a proper ideal of the normaliser of the Lie subalgebra.

Proposition 3.3.18
A Cartan subalgebra of a Lie algebra is a maximal nilpotent Lie subalgebra.

PROPOSITION 3.3.19
Suppose that g is a finite dimensional Lie algebra over an algebraically closed field and let x be its regular element contained in a Cartan subalgebra h. We have $h = g_0^x$.

Proof. The Cartan subalgebra h is contained in the Cartan subalgebra g_0^x by Engel's theorem. We have $h = g_0^x$ by Proposition 3.3.18. □

PROPOSITION 3.3.20
Suppose that h is a nilpotent Lie subalgebra of a finite dimensional Lie algebra g over an algebraically closed field of characteristic 0.

1. We have the weight space decomposition

$$g = \bigoplus_{\mu \in (h/Dh)^*} g_\mu^h, \quad g_\mu^h = \bigcap_{x \in h} g_{\mu(x)}^x, \quad g_{\mu(x)}^x = \lim_{n \to \infty} \ker\bigl(\operatorname{ad} x - \mu(x)\bigr)^n.$$

2. The set $\left[g_\mu^h, g_\nu^h\right]$ is contained in the subspace $g_{\mu+\nu}^h$ for any one dimensional representations μ and ν.

3. The nilpotent Lie algebra h is a Lie subalgebra of the Lie algebra g_0^h.

Proof. 3. By Engel's theorem. □

PROPOSITION 3.3.21
Suppose that h is a Cartan subalgebra of a finite dimensional Lie algebra g over an algebraically closed field of characteristic 0. We have $h = g_0^h$.

Proof. We have

$$\operatorname{ad}_{g_0^h} x = \begin{pmatrix} \operatorname{ad}_h x & * \\ 0 & \operatorname{ad}_{g_0^h/h} x \end{pmatrix}$$

for any element x of the Cartan subalgebra h. The element $\operatorname{ad}_{g_0^h/h} x$ is nilpotent for any element x of the Cartan subalgebra h since the element $\operatorname{ad}_{g_0^h} x$ is nilpotent. Suppose that the Cartan subalgebra h is a proper Lie subalgebra of the Lie algebra g_0^h. We have

$$\bigcap_{x \in h} \ker \operatorname{ad}_{g_0^h/h} x \neq \{0\}$$

by Engel's theorem. This is a contradiction. □

PROPOSITION 3.3.22
Suppose that h is a nilpotent Lie subalgebra of a finite dimensional Lie algebra g over an algebraically closed field of characteristic 0. We have

$$g_{\mu_1}^h \perp g_{\mu_2}^h$$

if we have $\mu_1 + \mu_2 \neq 0$.

Proof. Suppose that (x_1, x_2) is an arbitrary element of the space $g^h_{\mu_1} \times g^h_{\mu_2}$. We have $B(x_1, x_2) = 0$ since the space $(\operatorname{ad} x_1)(\operatorname{ad} x_2)g^h_\mu$ is contained in the space $g^h_{\mu+\mu_1+\mu_2}$ for any weight μ. □

COROLLARY 3.3.2
Suppose that h is a nilpotent Lie subalgebra of a finite dimensional semisimple Lie algebra g over an algebraically closed field of characteristic 0. The restriction of the Killing form to the space $g^h_0 \times g^h_0$ is nondegenerate.

COROLLARY 3.3.3
Suppose that h is a Cartan subalgebra of a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0. The restriction of the Killing form to the space $h \times h$ is nondegenerate.

THEOREM 3.3.3
Suppose that h is a Cartan subalgebra of a finite dimensional Lie algebra g over an algebraically closed field of characteristic 0. The set $(\operatorname{Inn} g)(h)$ has nonempty Zariski interior.

Proof. Suppose that ν is a nonzero weight and let x be an element of the space g^h_ν. We show that the derivation $\operatorname{ad} x$ is nilpotent. We have $(\operatorname{ad} x)^n g^h_\mu = \{0\}$ eventually for any weight μ since we have $g^h_{\mu+n\nu} = \{0\}$ eventually.

We write

$$h^\perp = \bigoplus_{\mu \text{ is a nonzero weight}} g^h_\mu$$

and let $(x_k)^n_{k=1}$ be a basis of the subspace $h^\perp$ contained in the set

$$\bigcup_{\mu \text{ is a nonzero weight}} g^h_\mu.$$

We define an element f of the space $S(g^*) \otimes g$ by

$$f(x_0 + \sum_{k=1}^n \nu_k x_k) = \exp(-\nu_1 \operatorname{ad} x_1) \cdots \exp(-\nu_n \operatorname{ad} x_n)(x_0)$$

for any element (x_0, ν) of the set $h \times \mathbb{F}^n$. The set $(\operatorname{Inn} g)(h)$ contains the image of the mapping f. Suppose that x is an element of the Cartan subalgebra h. We have

$$f'(x) = \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{ad} x \end{pmatrix}$$

since we have

$$\begin{aligned} f'(x)\begin{pmatrix} x_0 \\ \sum_{k=1}^n \nu_k x_k \end{patrix} &= \left(\frac{d}{dt}\right)_{t=0} f\big(x + t(x_0 \oplus \sum_{k=1}^n \nu_k x_k)\big) \\ &= x_0 - \sum_{k=1}^n \nu_k (\operatorname{ad} x_k)(x) \\ &= x_0 \oplus (\operatorname{ad} x)(\sum_{k=1}^n \nu_k x_k) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{ad} x \end{pmatrix} \begin{pmatrix} x_0 \\ \sum_{k=1}^n \nu_k x_k \end{pmatrix} \end{aligned}$$

for any element $x_0 \oplus \sum_{k=1}^n \nu_k x_k$ of the Lie algebra $g = h \oplus h^\perp$. We have $\det f'(x) \neq 0$ if and only if the element x belongs to the set

$$h \setminus \bigcup_{\mu \text{ is a nonzero weight}} \ker \mu. \tag{3.5}$$

The set (3.5) is not empty by Proposition 3.2.22. The image of the mapping f has nonempty Zariski interior by Corollary 3.2.2. $\square$

Theorem 3.3.4 (Chevalley)
The group of inner automorphisms of a finite dimensional Lie algebra over an algebraically closed field of characteristic 0 acts on the set of Cartan subalgebras transitively.

Proof. The Lie algebra g has a Cartan subalgebra by Corollary 3.3.1. Suppose that h_1 and h_2 are arbitrary Cartan subalgebras of the Lie algebra g. The set

$$(\operatorname{Inn} g)(h_1) \cap \big\{ \text{regular elements of the Lie algebra } g \big\} \cap (\operatorname{Inn} g)(h_2)$$

is not empty by Proposition 3.3.11 and Theorem 3.3.3. $\square$

Proposition 3.3.23
[1] Suppose that h is an ideal of a finite dimensional nilpotent Lie algebra g. We have $h = \{0\}$ if we have $h \cap \ker \operatorname{ad}_g = \{0\}$.

Proposition 3.3.24
Any finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0 has a maximal toral subalgebra.

Proposition 3.3.25
Suppose that $x = S + N$ is a Jordan decomposition of a linear mapping on a finite dimensional vector space over an algebraically closed field. We have $\ker x = \ker S \cap \ker N$.

Proof. By Corollary 2.3.2. $\square$

[1] [3, Lemma 3.3].

PROPOSITION 3.3.26
Suppose that h is a toral subalgebra of a finite dimensional semisimple Lie algebra g over an algebraically closed field of characteristic 0. We have the weight space decomposition

$$g = \bigoplus_{\mu \in h^*} g^h_\mu, \qquad g^h_\mu = \bigcap_{x \in h} g^x_{\mu(x)}, \qquad g^x_{\mu(x)} = \ker\big(\operatorname{ad} x - \mu(x)\big).$$

THEOREM 3.3.5
[2] A Lie subalgebra of a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0 is Cartan if and only if it is maximal toral.

COROLLARY 3.3.4
A Lie subalgebra of a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0 is Cartan if and only if it is toral and maximal abelian.

COROLLARY 3.3.5
Suppose that h is a Cartan subalgebra of a finite dimensional semisimple Lie algebra g over an algebraically closed field of characteristic 0.

1. We have the weight space decomposition

$$g = h \oplus \bigoplus_{\mu \in h^* \setminus \{0\}} g_\mu, \qquad g_\mu = \bigcap_{x \in h} \ker\big(\operatorname{ad} x - \mu(x)\big).$$

2. We have

$$h = g_0 = \bigcap_{x \in h} \ker(\operatorname{ad} x).$$

3.4 Lie Algebras $M(2,\mathbb{F}) \cap \ker \operatorname{tr}$

DEFINITION 3.4.1
Suppose that $\mathbb{F}$ is a field and let $(e^n)_{n=1}^d$ be the dual basis of the canonical basis $(e_n)_{n=1}^d$ of the vector space $\mathbb{F}^d$. We write $e_i^j = e_i \otimes e^j$ for any (i,j).

PROPOSITION 3.4.1
Suppose that $\mathbb{F}$ is a field and let d be a positive integer. The mapping

$$h \mapsto \sum_{n=1}^{d} (d-2n+1) e_n^n, \quad x \mapsto \sum_{n=1}^{d-1} (d-n) e_n^{n+1}, \quad y \mapsto \sum_{n=1}^{d-1} n e_{n+1}^n \tag{3.6}$$

into the Lie algebra $M(d,\mathbb{F})$ defines a representation of the Lie algebra

$$M(2,\mathbb{F}) \cap \ker \operatorname{tr}.$$

[2] [3, Proposition 8.2].

Remark 3.4.1
We have

$$[h,x]=2x, \qquad [h,y]=-2y, \qquad [x,y]=h.$$

Proof. We define

$$f(h)=\sum_{n=1}^{d}(d-2n+1)e_n^n, \quad f(x)=\sum_{n=1}^{d-1}(d-n)e_n^{n+1}, \quad f(y)=\sum_{n=1}^{d-1}ne_{n+1}^n.$$

It is sufficient to show that we have

$$\big[f(h),f(x)\big]=2f(x), \quad \big[f(h),f(y)\big]=-2f(y), \quad \big[f(x),f(y)\big]=f(h).$$

We have

$$\begin{aligned}\big[f(h),f(x)\big] &= \Big[\sum_{n=1}^{d}(d-2n+1)e_n^n,\sum_{n=1}^{d-1}(d-n)e_n^{n+1}\Big]\\ &=\sum_{n=1}^{d-1}(d-2n+1)(d-n)e_n^{n+1}-\sum_{n=1}^{d-1}(d-2n-1)(d-n)e_n^{n+1}\\ &=2\sum_{n=1}^{d-1}(d-n)e_n^{n+1}=2f(x).\end{aligned}$$

We have

$$\begin{aligned}\big[f(h),f(y)\big] &= \Big[\sum_{n=1}^{d}(d-2n+1)e_n^n,\sum_{n=1}^{d-1}ne_{n+1}^n\Big]\\ &=\sum_{n=1}^{d-1}(d-2n-1)ne_{n+1}^n-\sum_{n=1}^{d-1}(d-2n+1)ne_{n+1}^n\\ &=-2\sum_{n=1}^{d-1}ne_{n+1}^n=-2f(y).\end{aligned}$$

We have

$$\begin{aligned}\big[f(x),f(y)\big] &= \Big[\sum_{n=1}^{d-1}(d-n)e_n^{n+1},\sum_{n=1}^{d-1}ne_{n+1}^n\Big]\\ &=\sum_{n=1}^{d-1}(d-n)n\Big(e_n^n-e_{n+1}^{n+1}\Big)\\ &=\sum_{n=1}^{d}(d-n)ne_n^n-\sum_{n=1}^{d}(d-n+1)(n-1)e_n^n\\ &=\sum_{n=1}^{d}(d-2n+1)e_n^n=f(h).\end{aligned}$$

□

Proposition 3.4.2
The representations (3.6) are irreducible if the field $\mathbb{F}$ is of characteristic 0.

Remark 3.4.2
Suppose that $\mathbb{F}$ is a field. The mapping

$$h \mapsto -h, \qquad x \mapsto y, \qquad y \mapsto x$$

defines an automorphism of the Lie algebra $M(2,\mathbb{F}) \cap \ker \mathrm{tr}$.

Proposition 3.4.3
Suppose that $\mathbb{F}$ is a field and let f be a homomorphism of the Lie algebra $M(2,\mathbb{F}) \cap \ker \mathrm{tr}$ into an algebra over the field $\mathbb{F}$. We have

$$\big[f(h), f(x)^n\big] = 2nf(x)^n, \qquad \big[f(h), f(y)^n\big] = -2nf(y)^n$$

for any positive integer n.

Proof. The proof is by induction on the positive integer n. Suppose that we have $n > 1$. We have

$$\big[f(h), f(x)^n\big] = \big[f(h), f(x)^{n-1}\big] f(x) + f(x)^{n-1}\big[f(h), f(x)\big] = 2nf(x)^n.$$

By the above remark we have $\big[f(-h), f(y)^n\big] = 2nf(y)^n$. □

Proposition 3.4.4
Suppose that $\mathbb{F}$ is a field and let f be a homomorphism of the Lie algebra $M(2,\mathbb{F}) \cap \ker \mathrm{tr}$ into an algebra with identity over the field $\mathbb{F}$. We have

$$\begin{aligned}\big[f(x), f(y)^n\big] &= nf(y)^{n-1}f(h) - n(n-1)f(y)^{n-1}\\ &= nf(h)f(y)^{n-1} + n(n-1)f(y)^{n-1}\end{aligned}$$

and we have

$$\begin{aligned}\big[f(y), f(x)^n\big] &= -nf(h)f(x)^{n-1} + n(n-1)f(x)^{n-1}\\ &= -nf(x)^{n-1}f(h) - n(n-1)f(x)^{n-1}\end{aligned}$$

for any positive integer n.

Proof. The proof is by induction on the positive integer n. Suppose that we have $n > 1$. We have

$$\big[f(x), f(y)^n\big] = \big[f(x), f(y)^{n-1}\big] f(y) + f(y)^{n-1}f(h).$$

We have

$$\big[f(x), f(y)^{n-1}\big] = (n-1)f(y)^{n-2}f(h) - (n-1)(n-2)f(y)^{n-2}$$

by the induction hypothesis. We have

$$\begin{aligned}f(y)^{n-2}f(h)f(y) &= f(y)^{n-1}f(h) + f(y)^{n-2}\big[f(h), f(y)\big]\\ &= f(y)^{n-1}f(h) - 2f(y)^{n-1}.\end{aligned}$$

We have $\big[f(x), f(y)^n\big] = nf(y)^{n-1}f(h) - n(n-1)f(y)^{n-1}$. □

PROPOSITION 3.4.5
Suppose that $\mathbb{F}$ is an algebraically closed field of characteristic 0 and let f be a finite dimensional irreducible representation of the Lie algebra $M(2,\mathbb{F})\cap\ker\operatorname{tr}$. An eigenvector of the endomorphism $f(h)$ contained in the kernel of the endomorphism $f(x)$ is called a primitive vector of the irreducible representation f. There exists a primitive vector of the irreducible representation f.

Proof. The endomorphism $f(h)$ has an eigenvector ξ with eigenvalue ν since the field $\mathbb{F}$ is algebraically closed. We have

$$\begin{aligned} f(h)f(x)^n\xi &= \nu f(x)^n\xi + \big[f(h), f(x)^n\big]\xi \\ &= (\nu+2n)f(x)^n\xi \end{aligned}$$

for any nonnegative integer n. We have eventually $f(x)^n\xi = 0$ since the field $\mathbb{F}$ is of characteristic 0. We define

$$n = \min\big\{\, n\in\mathbb{N} : f(x)^n\xi = 0 \,\big\}.$$

The vector $f(x)^{n-1}\xi$ is a primitive vector of the irreducible representation f. □

THEOREM 3.4.1
Suppose that $\mathbb{F}$ is an algebraically closed field of characteristic 0. A finite dimensional irreducible representation of the Lie algebra $M(2,\mathbb{F})\cap\ker\operatorname{tr}$ is equivalent to the irreducible representation (3.6).

Proof. There exists a primitive vector e of the irreducible representation f by the previous proposition. We define

$$e_n = \frac{1}{(n-1)!}f(y)^{n-1}e$$

for any positive integer n. □

3.5 Root Systems

PROPOSITION 3.5.1
Suppose that $\mathbb{F}$ is a field.

1. We have

$$\big\{\text{ monic polynomials }\big\} \simeq \big\{\text{ nonzero ideals of } \mathbb{F}[x]\,\big\}, \quad f(x)\mapsto \mathbb{F}[x]f(x).$$

2. We have

$$\begin{aligned} \big\{\text{ monic irreducible polynomials }\big\} &\simeq \big\{\text{ nonzero prime ideals of } \mathbb{F}[x]\,\big\} \\ &= \big\{\text{ maximal ideals of } \mathbb{F}[x]\,\big\}. \end{aligned}$$

PROPOSITION 3.5.2
Suppose that $\mathbb{F}$ is an algebraically closed field.

1. Suppose that $f(x)$ is an element of $\mathbb{F}[x]$. We have
$$f(x) \text{ is an irreducible polynomial} \Leftrightarrow \deg f(x) = 1.$$
2. We have
$$\mathbb{F} \simeq \{\text{ monic irreducible polynomials }\}, \qquad \nu \mapsto x - \nu.$$

Proposition 3.5.3
Suppose that R is a commutative ring with identity.

1. A unital module over the algebra $R[x]$ is a unital module over the ring R.
2. Suppose that M is a unital module over the ring R. We have
$$\hom M \simeq \{\text{ compatible unital module structures over the algebra } R[x] \text{ on the unital module } M\}.$$

Proposition 3.5.4
Suppose that x and y are elements of a commutative ring with identity. We have $(x)(y) = (xy)$.

Proposition 3.5.5
Suppose that x and y are elements of an integral domain. We have $(x) = (y)$ if and only if we have $R^\times x = R^\times y$.

Definition 3.5.1
An element x of a commutative ring with identity is called a prime element if the principal ideal (x) is a nonzero prime ideal.

Theorem 3.5.1
We have
$$\{\text{ nonzero ideals }\} = \mathbb{Z}_+^{\oplus\{\text{ nonzero prime ideals }\}}$$
for a principal ideal domain.

Definition 3.5.2
Suppose that M is a unital module over an integral domain R. We define
$$t(M) = M \cap \Big\{ x : \text{the element } 0 \text{ is contained in the set } \big(R \setminus \{0\}\big)x \Big\}.$$

Proposition 3.5.6
Suppose that R is a principal ideal domain and let
$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$
be an element of the set $R^2 \setminus \{0\}$. There exists a greatest common divisor d of the set $\{r_1, r_2\}$ by Proposition 2.3.5. There exists a basis (e_1, e_2) of the free module R^2 such that we have
$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = de_1.$$

Proof. We define

$$e_1 = d^{-1}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

There exists an element e_2 of the free module R^2 such that we have

$$\det(e_1, e_2) = 1.$$

The pair (e_1, e_2) is a basis of the free module R^2 and we have

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = de_1. \quad \square$$

THEOREM 3.5.2
Suppose that R is a principal ideal domain and let M be a submodule of a free module R^n. There exists a unique decreasing sequence $(R_k)_{k=1}^m$ of nonzero ideals such that there exists a basis $(e_k)_{k=1}^n$ of the free module R^n with $M = \bigoplus_{k=1}^m R_k e_k$.

Proof. The proof is by induction on n.

Suppose that we have $n > 0$. We may assume that we have $M \neq \{0\}$. We define an ideal

$$R_1\big((e_k)_{k=1}^n\big) = \Big\{\, r_1 \in R : M \cap (r_1 e_1 + \bigoplus_{k=2}^n Re_k) \text{ is not empty} \,\Big\}$$

for a basis $(e_k)_{k=1}^n$ of the module R^n. The set

$$\Big\{\, R_1\big((e_k)_{k=1}^n\big) : (e_k)_{k=1}^n \text{ is a basis of the module } R^n \,\Big\} \tag{3.7}$$

has a maximal element

$$(r_1) = R_1 = R_1(e_1, \dots, e_n) \neq \{0\}$$

since the domain R is Noetherian.

There exists an element $(r_k)_{k=2}^n$ of the module R^{n-1} such that the element

$$r_1 e_1 + r_2 e_2 + \cdots + r_n e_n$$

belongs to the submodule M.

We proceed to show that the set $\{r_k\}_{k=2}^n$ is contained in the ideal (r_1). There exists a greatest common divisor r of the set $\{r_1, r_2\}$ by Proposition 2.3.5. There exists an element P of the group $M(2, R)^\times$ such that we have

$$r_1 e_1 + r_2 e_2 = r(P_{11} e_1 + P_{21} e_2)$$

by Proposition 3.5.6. The ideal (r) is contained in the ideal

$$R_1\big((e_1, e_2)P, e_3, \dots, e_n\big)$$

and we have $(r_1) = (r)$ since the ideal (r_1) is a maximal element of the set (3.7).

We may assume that the module $R_1 e_1$ is contained in the module M since we have

$$R_1 = R_1\left(e_1 + \frac{r_2}{r_1}e_2 + \cdots + \frac{r_n}{r_1}e_n, e_2, \ldots, e_n\right).$$

We have

$$M = R_1 e_1 \oplus (M \cap \bigoplus_{k=2}^{n} Re_k).$$

We may assume that there exists a decreasing sequence $(R_k)_{k=2}^m$ of nonzero ideals such that we have $M = \bigoplus_{k=1}^m R_k e_k$ by the induction hypothesis. The sequence $(R_k)_{k=1}^m$ is decreasing. □

Corollary 3.5.1

Suppose that R is a principal ideal domain. A submodule of a free module R^n is a free module whose rank is less than n.

Theorem 3.5.3

Suppose that M is a finitely generated unital module over a principal ideal domain R.

1. There exists a unique decreasing sequence $(R_k)_{k=1}^m$ of nonzero proper ideals such that we have
$$M = \bigoplus_{k=1}^{m} R/R_k \oplus R^{n-m}.$$
2. We have
$$t(M) = \bigoplus_{k=1}^{m} R/R_k.$$

Theorem 3.5.4

Suppose that M is a finitely generated unital module over a principal ideal domain R.

1. There exists a unique element (m, n) of the set
$$\mathbb{Z}_+^{\oplus(\{\text{ nonzero prime ideals }\}\times\mathbb{N})} \times \mathbb{Z}_+$$
such that we have
$$M = \bigoplus_{P} \bigoplus_{n=1}^{\infty} (R/P^n)^{m(P,n)} \oplus R^n.$$
2. We have
$$\lim_{n\to\infty} M \cap \{\, x : P^n x = \{0\} \,\} = \bigoplus_{n=1}^{\infty} (R/P^n)^{m(P,n)}$$
for $\forall P$.

3. We have

$$\begin{aligned} t(M) &= \bigoplus_P \lim_{n\to\infty} M \cap \{\, x : P^n x = \{0\} \,\} \\ &= \bigoplus_P \bigoplus_{n=1}^{\infty} (R/P^n)^{m(P,n)}. \end{aligned}$$

Proof. We have

$$\lim_{n\to\infty} M \cap \{\, x : P^n x = \{0\} \,\} = \bigoplus_{n=1}^{\infty} (R/P^n)^{m(P,n)}$$

for $\forall P$. We may assume that

$$M = \bigoplus_{k=1}^{m} R/P^{n_k}$$

for some $1 \leq n_1 \leq \cdots \leq n_m$. The proof is by induction on

$$\min\{\, n : P^n M = \{0\} \,\}.$$

Suppose that we have

$$\min\{\, n : P^n M = \{0\} \,\} > 0.$$

We have

$$PM = \bigoplus_{k=1}^{m} R/P^{n_k - 1}.$$

The module $M/PM = (R/P)^m$ is a vector space over a field R/P. □

Theorem 3.5.5
Suppose that $\mathbb{F}$ is an algebraically closed field and let V be a unital module over the algebra $\mathbb{F}[x]$ whose dimension over the field $\mathbb{F}$ is finite.

1. There exists a unique element m of the set $\mathbb{Z}_+^{\oplus(\mathbb{F}\times\mathbb{N})}$ such that we have

$$V = \bigoplus_{\nu\in\mathbb{F}} \bigoplus_{n=1}^{\infty} \left(\frac{\mathbb{F}[x]}{\mathbb{F}[x](x-\nu)^n} \right)^{m(\nu,n)}.$$

2. We have

$$\lim_{n\to\infty} \ker(x-\nu)^n = \bigoplus_{n=1}^{\infty} \left(\frac{\mathbb{F}[x]}{\mathbb{F}[x](x-\nu)^n} \right)^{m(\nu,n)}$$

for $\forall \nu$.

3. The finite sequence
$$\left((x-\nu)^{n-k}\right)_{k=1}^{n}$$
is a basis of the vector space $\dfrac{\mathbb{F}[x]}{\mathbb{F}[x](x-\nu)^n}$ over the field $\mathbb{F}$ and we have
$$x\left((x-\nu)^{n-k}\right)_{k=1}^{n} = \left((x-\nu)^{n-k}\right)_{k=1}^{n} \left(\nu + \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & \end{pmatrix}\right)$$
for $\forall(\nu, n)$.

Theorem 3.5.6 (Hamilton-Cayley)

Suppose that x is a linear mapping on a finite dimensional vector space over a field and let f be its characteristic polynomial. We have $f(x) = 0$.

Proof. We may assume that the field is algebraically closed and we have
$$x = \nu + \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & \end{pmatrix}$$
for some element ν of the field by Theorem 3.5.5. We have
$$f(x) = (x-\nu)^n = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & \end{pmatrix}^n = 0. \quad \square$$

Definition 3.5.3

Suppose that $\mathbb{F}$ is a field of characteristic 0 and let H be a finite dimensional vector space over the field $\mathbb{F}$. Suppose that σ is a linear mapping on the vector space H and let ξ be an element of the set $H \setminus \{0\}$. The linear mapping σ is called a reflection along the element ξ if we have
$$\operatorname{codim}\ker(\sigma - 1) = 1, \qquad (\sigma+1)(\xi) = 0.$$

Proposition 3.5.7

Suppose that the linear mapping σ is a reflection along the element ξ. We have
$$H = \ker(\sigma - 1) \oplus \mathbb{F}\xi$$
and we have $\sigma^2 = 1$.

Proposition 3.5.8

Suppose that $\mathbb{F}$ is a field of characteristic 0 and let H be a finite dimensional vector space over the field $\mathbb{F}$. Suppose that Δ is a finite subset of the space H with $H = \operatorname{span}\Delta$ and let ξ be an element of the set $H \setminus \{0\}$. There exists at most one reflection σ_ξ along the element ξ with $\sigma_\xi(\Delta) = \Delta$.

Proof. Suppose that σ_1 and σ_2 are reflections along the element ξ with $\sigma_1(\Delta) = \sigma_2(\Delta) = \Delta$ and let $\sigma = \sigma_1 \circ \sigma_2$. We have $\sigma^n = 1$ for some positive integer n since we have $\sigma(\Delta) = \Delta$. The minimal polynomial $f(x)$ of the linear mapping σ does not have a multiple root since it divides the polynomial $x^n - 1$. We have $f(x) = x - 1$ since the polynomial $(x-1)^{\dim H}$ is the characteristic polynomial of the linear mapping σ. $\square$

Definition 3.5.4

Suppose that $\mathbb{F}$ is a field of characteristic 0 and let H be a finite dimensional vector space over the field $\mathbb{F}$. A finite subset Δ of the set $H \setminus \{0\}$ is called a root system if it satisfies the following.

1. We have $H = \operatorname{span} \Delta$.
2. There exists a reflection σ_ξ along the element ξ with $\sigma_\xi(\Delta) = \Delta$ for any element ξ of the set Δ.
3. The element $\eta - \sigma_\xi(\eta)$ belongs to the set $\mathbb{Z}\xi$ for any element (ξ, η) of the set Δ^2.

Proposition 3.5.9

Suppose that ξ is an element of a root system Δ. There exists a unique element ξ^* of the vector space $(\operatorname{span} \Delta)^*$ such that we have $\sigma_\xi = 1 - \xi \otimes \xi^*$.

Proposition 3.5.10

Suppose that ξ is an element of a root system. We have $\xi^*(\xi) = 2$.

Proposition 3.5.11

Suppose that ξ and η are elements of a root system. The element $\xi^*(\eta)$ is an integer.

Proposition 3.5.12

Suppose that ξ is an element of a root system Δ. We have

$$\Delta \cap \langle \xi \rangle = \{\pm\xi\}, \left\{\pm\frac{\xi}{2}, \pm\xi\right\}, \{\pm\xi, \pm 2\xi\}.$$

Proof. Suppose that x is a scalar such that the element $x\xi$ belongs to the root system Δ. The scalars $2x$ and $2x^{-1}$ are integers since the element

$$2x\xi = x\xi - \sigma_\xi(x\xi)$$

belongs to the set $\mathbb{Z}\xi$. We have

$$x = \pm\frac{1}{2}, \pm 1, \pm 2. \quad \square$$

Definition 3.5.5

A root system Δ is said to be reduced if we have

$$\Delta \cap \langle \xi \rangle = \{\pm\xi\}$$

for any element ξ of the root system Δ.

Definition 3.5.6

Suppose that Δ is a root system. The subgroup of the finite group

$$(\operatorname{hom}\operatorname{span}\Delta)^{\times} \cap \left\{ \text{permutations of the root system } \Delta \right\}$$

generated by the set

$$\left\{ \sigma_{\xi} : \xi \text{ is an element of the root system } \Delta \right\}$$

is called the Weyl group and denoted by $W(\Delta)$.

Proposition 3.5.13

Suppose that Δ is a real root system. There exists an inner product on the real vector space $\operatorname{span}\Delta$ such that we have

$$\big\|\sigma(\xi)\big\| = \|\xi\|$$

for any element (σ, ξ) of the set $W(\Delta) \times \operatorname{span}\Delta$.

Proof. Suppose that

$$(\operatorname{span}\Delta)^2 \to \mathbb{R}, \qquad (\xi, \eta) \mapsto (\xi, \eta)'$$

is an arbitrary inner product on the real vector space $\operatorname{span}\Delta$. The mapping

$$(\operatorname{span}\Delta)^2 \to \mathbb{R}, \qquad (\xi, \eta) \mapsto (\xi, \eta) = \sum_{\sigma \in W(\Delta)} \big(\sigma(\xi), \sigma(\eta)\big)'$$

is an inner product on the real vector space $\operatorname{span}\Delta$ such that we have

$$\sqrt{\big(\sigma(\xi), \sigma(\xi)\big)} = \sqrt{(\xi, \xi)}$$

for any element (σ, ξ) of the set $W(\Delta) \times \operatorname{span}\Delta$. □

Proposition 3.5.14

Suppose that Δ is a real root system. We have

$$\xi^*(\eta) = \frac{2(\xi, \eta)}{\|\xi\|^2}, \qquad \sigma_{\xi}(\eta) = \eta - \frac{2(\xi, \eta)}{\|\xi\|^2}\xi$$

for any elements (ξ, η) of the set $\Delta \times \operatorname{span}\Delta$.

Proof. We have

$$\begin{aligned}(\xi, \eta) &= \big(\sigma_{\xi}(\xi), \sigma_{\xi}(\eta)\big) \\ &= \big(-\xi, \eta - \xi^*(\eta)\xi\big) = -(\xi, \eta) + \xi^*(\eta)\|\xi\|^2.\end{aligned}$$ □

Proposition 3.5.15

Suppose that Δ is a real root system. The set

$$\Delta^* = \left\{ \xi^* = \frac{2\xi}{\|\xi\|^2} : \xi \text{ is an element of the root system } \Delta \right\}$$

is a real root system.

Proof. The set Δ^* is a finite subset of the set $(\operatorname{span}\Delta)^* \setminus \{0\}$.

1. We have $(\operatorname{span}\Delta)^* = \operatorname{span}\Delta^*$ by Proposition 3.5.14.

2. Suppose that ξ is an element of the root system Δ. The linear mapping
$$1 - \xi^* \otimes \xi$$
is a reflection along the element ξ^*. We have
$$(1 - \xi^* \otimes \xi)(\Delta^*) = \Delta^*$$
since we have
$$\begin{aligned}(1 - \xi^* \otimes \xi)(\eta^*)(\zeta) &= \eta^*(\zeta) - \eta^*(\xi)\xi^*(\zeta) \\ &= \frac{2(\eta,\zeta)}{\|\eta\|^2} - \frac{4(\xi,\eta)(\xi,\zeta)}{\|\xi\|^2\|\eta\|^2} \\ &= \frac{2\big(\sigma_\xi(\eta),\zeta\big)}{\big\|\sigma_\xi(\eta)\big\|^2} = \sigma_\xi(\eta)^*(\zeta)\end{aligned}$$
for any element (η,ζ) of the set $\Delta \times \operatorname{span}\Delta$.

3. The element
$$\eta^* - \sigma_{\xi^*}(\eta^*) = \eta^*(\xi)\xi^*$$
belongs to the set $\mathbb{Z}\xi^*$ for any element (ξ,η) of the set Δ^2. □

Proposition 3.5.16
Suppose that Δ is a real root system.

1. Suppose that ξ is an element of the root system Δ. We have $\xi^{**} = \xi$.

2. We have $\#\Delta = \#\Delta^*$.

Proposition 3.5.17
Suppose that Δ is a reduced real root system. The real root system Δ^* is reduced.

Proposition 3.5.18
Suppose that V is a finite dimensional vector space over a subfield of a field $\mathbb{F}$. We have $(V \otimes \mathbb{F})^* = V^* \otimes \mathbb{F}$.

Theorem 3.5.7
Suppose that Δ is a complex root system. We have $\operatorname{span}\Delta = \operatorname{span}_{\mathbb{R}}\Delta \otimes \mathbb{C}$ and the set Δ is a real root system.

Proof. We show that the set Δ is a real root system. Suppose that ξ is an element of the root system Δ. The real vector space $\operatorname{span}_{\mathbb{R}}\Delta$ is invariant for the real linear mapping σ_ξ since we have $\sigma_\xi(\Delta) = \Delta$. We show that the restriction of the real linear mapping σ_ξ to the invariant subspace $\operatorname{span}_{\mathbb{R}} \Delta$ is a reflection along the element ξ. It is sufficient to show that

$$\dim \frac{\operatorname{span}_{\mathbb{R}} \Delta}{\ker(\sigma_\xi - 1) \cap \operatorname{span}_{\mathbb{R}} \Delta} = 1.$$

Suppose that η is an element of the root system Δ. There exists a complex number z such that the element $\eta - z\xi$ belongs to the set $\ker(\sigma_\xi - 1)$ since we have $\operatorname{span} \Delta = \ker(\sigma_\xi - 1) \oplus \mathbb{C}\xi$. The complex number z is real since we have $\xi^*(\eta - z\xi) = 0$. The element η belongs to the set $\big(\ker(\sigma_\xi - 1) \cap \operatorname{span}_{\mathbb{R}} \Delta\big) \oplus \mathbb{R}\xi$.

We show that we have $\operatorname{span} \Delta = \operatorname{span}_{\mathbb{R}} \Delta \otimes \mathbb{C}$. There exists a complex linear mapping f of the space $\operatorname{span}_{\mathbb{R}} \Delta \otimes \mathbb{C}$ onto the space $\operatorname{span} \Delta$ with $f(\xi \otimes z) = z\xi$ for each (ξ, z). It is sufficient to show that the dual mapping of the mapping f is surjective. The dual mapping f^* is surjective since we have $f^*(\xi^*) = \xi^* \otimes 1$ for any element ξ of the root system Δ and the set Δ^* is a real root system. $\square$

Proposition 3.5.19
Any root system is a root system over any subfield.

Proposition 3.5.20
Any root system is a root system over any extension field.

Proposition 3.5.21
Suppose that Δ is a real root system and let ξ and η be elements of the root system Δ such that we have $\mathbb{R}\xi \cap \mathbb{R}\eta = \{0\}$. We have

$\eta^*(\xi)$	$\xi^*(\eta)$	$\arccos \dfrac{(\xi, \eta)}{\lVert\xi\rVert \lVert\eta\rVert}$	
0	0	$\frac{\pi}{2}$	
1	1	$\frac{\pi}{3}$	$\lVert\xi\rVert = \lVert\eta\rVert$
-1	-1	$\frac{2\pi}{3}$	$\lVert\xi\rVert = \lVert\eta\rVert$
2	1	$\frac{\pi}{4}$	$\lVert\xi\rVert = \sqrt{2}\lVert\eta\rVert$
-2	-1	$\frac{3\pi}{4}$	$\lVert\xi\rVert = \sqrt{2}\lVert\eta\rVert$
3	1	$\frac{\pi}{6}$	$\lVert\xi\rVert = \sqrt{3}\lVert\eta\rVert$
-3	-1	$\frac{5\pi}{6}$	$\lVert\xi\rVert = \sqrt{3}\lVert\eta\rVert$

provided that we have $\big|\eta^*(\xi)\big| \geq \big|\xi^*(\eta)\big|$.

Proof. We have

$$\eta^*(\xi)\xi^*(\eta) = \frac{4(\xi, \eta)^2}{\lVert\xi\rVert^2 \lVert\eta\rVert^2} = 0, 1, 2, 3. \quad \square$$

Proposition 3.5.22
Suppose that Δ is a real root system and let ξ and η be elements of the root system Δ such that we have $\mathbb{R}\xi \cap \mathbb{R}\eta = \{0\}$. The element $\xi + \eta$ belongs to the root system Δ provided that we have $(\xi, \eta) < 0$.

Proof. We may assume that we have $\xi^*(\eta) = -1$. The element

$$\sigma_\xi(\eta) = -\xi^*(\eta)\xi + \eta = \xi + \eta$$

belongs to the root system Δ. $\square$

Definition 3.5.7
A subset S of a root system Δ is called a basis if it satisfies the following.

1. The set S is a linear basis of the vector space $\operatorname{span}\Delta$.
2. The root system Δ is contained in the set

$$\sum_{\xi \in S} \mathbb{Z}_+ \xi \cup \sum_{\xi \in S} \mathbb{Z}_- \xi.$$

Definition 3.5.8
Suppose that S is a basis of a root system Δ. We define

$$\Delta^+ = \sum_{\xi \in S} \mathbb{Z}_+ \xi \cap \Delta.$$

Proposition 3.5.23
Suppose that S is a basis of a root system Δ. We have $\Delta = \Delta^+ \sqcup (-\Delta^+)$.

Proposition 3.5.24
Suppose that S is a subset of a real inner product space H such that we have $\xi = \eta$ provided that ξ and η are elements of the subset S with $(\xi, \eta) > 0$. Suppose that f is a linear functional on the real vector space H such that the set $f(S)$ is contained in the set $(0, \infty)$. The subset S is linearly independent.

Proof. Suppose that $\big((x_k, \xi_k)\big)_{k=1}^{m+n}$ is an element of the set $\big((0,\infty) \times S\big)^{m+n}$ with $\#\big\{\xi_k\big\}_{k=1}^{m+n} = m + n$ and let

$$\xi = \sum_{k=1}^{m} x_k \xi_k = \sum_{k=m+1}^{m+n} x_k \xi_k.$$

We have $\xi = 0$ since we have

$$(\xi, \xi) = \sum_{i=1}^{m} \sum_{j=m+1}^{m+n} x_i x_j (\xi_i, \xi_j) \leq 0.$$

We have $m + n = 0$ since we have

$$0 = f(\xi) = \sum_{k=1}^{m} x_k f(\xi_k) = \sum_{k=m+1}^{m+n} x_k f(\xi_k). \quad \square$$

Proposition 3.5.25
Suppose that Δ is a real root system. There exists a linear functional f on the space $\operatorname{span}\Delta$ such that the set $f(\Delta)$ does not contain the element 0.

Definition 3.5.9
Suppose that Δ is a real root system and let f be a linear functional on the space $\operatorname{span}\Delta$ such that the set $f(\Delta)$ does not contain the element 0. We define

$$\Delta_f^+ = f^{-1}\big((0,\infty)\big) \cap \Delta, \qquad\qquad S_f = \Delta_f^+ \setminus (\Delta_f^+ + \Delta_f^+).$$

Proposition 3.5.26
Suppose that Δ is a real root system and let f be a linear functional on the space $\operatorname{span}\Delta$ such that the set $f(\Delta)$ does not contain the element 0. We have $\Delta = \Delta_f^+ \sqcup (-\Delta_f^+)$.

Proposition 3.5.27
Suppose that Δ is a real root system and let f be a linear functional on the real vector space $\operatorname{span}\Delta$ such that the set $f(\Delta)$ does not contain the element 0. We have $\xi = \eta$ provided that ξ and η are elements of the set S_f with $(\xi, \eta) > 0$.

Proof. Suppose that we have $\mathbb{R}\xi = \mathbb{R}\eta$. We have $\xi = \eta$ by Proposition 3.5.12. Suppose that we have $\mathbb{R}\xi \cap \mathbb{R}\eta = \{0\}$. The element $\zeta = \xi - \eta$ belongs to the set $\Delta \setminus \Delta_f^+$ by Proposition 3.5.22. This is a contradiction. □

Proposition 3.5.28
Suppose that Δ is a real root system and let f be a linear functional on the real vector space $\operatorname{span}\Delta$ such that the set $f(\Delta)$ does not contain the element 0. The set S_f is linearly independent.

Proof. By Proposition 3.5.24 and Proposition 3.5.27. □

Theorem 3.5.8
Suppose that Δ is a real root system and let f be a linear functional on the real vector space $\operatorname{span}\Delta$ such that the set $f(\Delta)$ does not contain the element 0. The set S_f is a basis of the real root system Δ and we have

$$\Delta_f^+ = \sum_{\xi \in S_f} \mathbb{Z}_+\xi \cap \Delta.$$

Proof. We show that the set Δ_f^+ is contained in the set

$$\sum_{\xi \in S_f} \mathbb{Z}_+\xi.$$

The proof is by induction on the number $f(\xi) > 0$. Suppose that ξ is an element of the set Δ_f^+ and we have

$$f(\xi) = \min_{\xi \in \Delta_f^+} f(\xi).$$

The element ξ belongs to the set S_f. Suppose that ξ is an element of the set Δ_f^+ and we have

$$f(\xi) > \min_{\xi \in \Delta_f^+} f(\xi).$$

We may assume that the element ξ belongs to the set $\Delta_f^+ + \Delta_f^+$. The element ξ belongs to the set

$$\sum_{\xi \in S_f} \mathbb{Z}_+ \xi$$

by the induction hypothesis. □

Corollary 3.5.2
Any real root system has a basis.

Proof. By Proposition 3.5.25. □

Proposition 3.5.29
Suppose that S is a basis of a real root system Δ. There exists a linear functional f on the real vector space $\operatorname{span}\Delta$ such that the set $f(S)$ is contained in the set $\{1\}$.

Proof. We define $n = \dim \operatorname{span} \Delta$. Suppose that $\{\delta^k\}_{k=1}^n$ is the dual basis of the linear basis S and let

$$f = \sum_{k=1}^n \delta^k.$$

The set $f(S)$ is contained in the set $\{1\}$. □

Proposition 3.5.30
Suppose that S is a basis of a real root system Δ and let f be a linear functional on the real vector space $\operatorname{span}\Delta$ such that the set $f(S)$ is contained in the set $(0, \infty)$. We have the following.

1. The set $f(\Delta)$ does not contain the element 0.
2. We have $\Delta^+ = \Delta_f^+$ and we have $S = S_f$.

Proof. The set Δ^+ is contained in the set Δ_f^+ since the basis S is contained in the set Δ_f^+. Suppose that ξ is an element of the set $\Delta_f^+ \setminus \Delta^+$. The element ξ belongs to the set $-\Delta_f^+$ since we have $\Delta = \Delta^+ \sqcup (-\Delta^+)$. This is a contradiction since we have $\Delta = \Delta_f^+ \sqcup (-\Delta_f^+)$. We have $\Delta^+ = \Delta_f^+$. The basis S is contained in the set $\Delta^+ \setminus (\Delta^+ + \Delta^+)$. The basis S is contained in the basis S_f since we have $\Delta^+ = \Delta_f^+$. We have $S = S_f$. □

Proposition 3.5.31
Suppose that S is a basis of a real root system Δ. We have $S = \Delta^+ \setminus (\Delta^+ + \Delta^+)$.

Proof. There exists a linear functional f on the real vector space $\operatorname{span}\Delta$ such that the set $f(S)$ is contained in the set $\{1\}$ by Proposition 3.5.29. We have $\Delta^+ = \Delta_f^+$ and we have $S = S_f$ by Proposition 3.5.30. □

PROPOSITION 3.5.32
Suppose that S is a basis of a real root system Δ. We have $\xi = \eta$ provided that ξ and η are elements of the basis S with $(\xi, \eta) > 0$.

Proof. There exists a linear functional f on the real vector space $\operatorname{span}\Delta$ such that the set $f(S)$ is contained in the set $\{1\}$ by Proposition 3.5.29. We have $S = S_f$ by Proposition 3.5.30. We have $\xi = \eta$ provided that ξ and η are elements of the basis S with $(\xi, \eta) > 0$ by Proposition 3.5.27. $\square$

PROPOSITION 3.5.33
Suppose that Δ is a real root system. We have

$$\begin{aligned}&\big\{\text{ bases of the real root system } \Delta \,\big\}\\ &\qquad = \big\{\, S_f : f \text{ is a linear functional on the real vector space } \operatorname{span}\Delta\\ &\qquad\qquad\qquad \text{such that the set } f(\Delta) \text{ does not contain the element } 0 \,\big\}.\end{aligned}$$

PROPOSITION 3.5.34
Suppose that S is a basis of a real root system Δ and let ξ be an element of the set Δ^+. There exists a finite sequence $\big\{\xi_k\big\}_{k=1}^n$ of elements of the basis S such that the elements

$$\xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \cdots + \xi_n = \xi$$

belong to the set Δ^+.

Proof. There exists a linear functional f on the real vector space $\operatorname{span}\Delta$ such that the set $f(S)$ is contained in the set $\{1\}$ by Proposition 3.5.29. The proof is by induction on the positive integer $f(\xi)$. Suppose that we have $f(\xi) = 1$. The element $\xi_1 = \xi$ belongs to the basis S. Suppose that we have $f(\xi) > 1$. The element ξ does not belong to the basis S and the set $S \cup \{\xi\}$ is linearly dependent. There exists an element ξ' of the real vector space $\operatorname{span}\Delta$ such that the element $\xi - \xi'$ belongs to the basis S and we have $(\xi, \xi - \xi') > 0$ by Proposition 3.5.24 and Proposition 3.5.32. We may assume that the element ξ does not belong to the set $\mathbb{R}(\xi - \xi')$. We have $f(\xi') = f(\xi) - 1 > 0$ and the element ξ' belongs to the set Δ^+ by Proposition 3.5.22. There exists a finite sequence $\big\{\xi_k\big\}_{k=1}^n$ of elements of the basis S such that the elements

$$\xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \cdots + \xi_n = \xi'$$

belong to the set Δ^+ by the induction hypothesis. We have

$$\xi_1 + \cdots + \xi_n + (\xi - \xi') = \xi. \quad \square$$

PROPOSITION 3.5.35
Suppose that S is a basis of a reduced real root system Δ. We have

$$\sigma_\xi\big(\Delta^+ \setminus \{\xi\}\big) = \Delta^+ \setminus \{\xi\}$$

for any element ξ of the basis S.

Proof. Suppose that we have $S = \{\xi_1, \dots, \xi_{n-1}, \xi\}$ and let $\{\delta^1, \dots, \delta^{n-1}, \delta\}$ be its dual basis. Suppose that η is an arbitrary element of the set $\Delta^+ \setminus \{\xi\}$. We have $\sum_{k=1}^{n-1} \delta^k(\eta) > 0$ since the real root system Δ is reduced. The element $\sigma_\xi(\eta)$ belongs to the set $\Delta^+ \setminus \{\xi\}$ since we have

$$\sum_{k=1}^{n-1} \delta^k\big(\sigma_\xi(\eta)\big) = \sum_{k=1}^{n-1} \delta^k\big(\eta - \xi^*(\eta)\xi\big) = \sum_{k=1}^{n-1} \delta^k(\eta) > 0. \quad \square$$

Proposition 3.5.36
Suppose that S is a basis of a reduced real root system Δ. We have

$$\sigma_\xi\Big(\sum \Delta^+\Big) = \sum \Delta^+ - 2\xi$$

for any element ξ of the basis S.

Definition 3.5.10
A subset C of a real vector space is called a cone if $(0, \infty)C = C$.

Proposition 3.5.37
Suppose that S is a subset of a real vector space. There exists the convex cone generated by the subset S.

Proposition 3.5.38
Suppose that $(\xi_k)_{k=1}^n$ is a finite sequence of a real vector space. The convex cone generated by the set $\{\xi_k\}_{k=1}^n$ is given by

$$\Big\{ \sum_{k=1}^n x_k \xi_k : (x_k)_{k=1}^n \text{ is an element of the set } [0, \infty)^n \setminus \{0\} \Big\}.$$

Proposition 3.5.39
Suppose that $(\xi_k)_{k=1}^n$ is an independent finite sequence of a real vector space and let C be the convex cone generated by the set $\{\xi_k\}_{k=1}^n$. We have

$$C = \bigsqcup_{k=1}^n (0, \infty)\xi_k \sqcup \bigcup_{m=2}^n \Big\{ \sum_{k=1}^m y_k \eta_k : (y_k)_{k=1}^m \text{ is an element of the set } (0, \infty)^m$$
$$\text{and } (\eta_k)_{k=1}^m \text{ is an independent finite sequence of the convex cone } C \Big\}.$$

Proposition 3.5.40
Suppose that S is a basis of a reduced real root system Δ. The linear basis

$$S^* = \{ \xi^* : \xi \text{ is an element of the basis } S \}$$

is a basis of the reduced real root system Δ^*.

Proof. There exists a linear functional f on the real vector space $\operatorname{span} \Delta = \operatorname{span} \Delta^*$ such that the set $f(S)$ is contained in the set $(0, \infty)$ by Proposition

3.5.29. We write C for the convex cone generated by the basis S. The convex cone C is generated by the set Δ^+. We have

$$(\Delta^*)^+_f = (\Delta^+)^* = \{\, \xi^* : \xi \text{ is an element of the set } \Delta^+ \,\}.$$

The convex cone C is generated by the basis

$$\begin{aligned} S^*_f &= (\Delta^*)^+_f \setminus \big((\Delta^*)^+_f + (\Delta^*)^+_f\big) \\ &= (\Delta^+)^* \setminus \big((\Delta^+)^* + (\Delta^+)^*\big) \end{aligned}$$

since it is generated by the set $(\Delta^*)^+_f = (\Delta^+)^*$. We have

$$\begin{aligned} &\{\, (0,\infty)\xi^* : \xi^* \text{ is an element of the linear basis } S^* \,\} \\ &\qquad = \{\, (0,\infty)\xi^* : \xi^* \text{ is an element of the basis } S^*_f \,\} \end{aligned}$$

since we have

$$\bigsqcup_{\xi^* \in S^*} (0,\infty)\xi^* = \bigsqcup_{\xi \in S} (0,\infty)\xi = \bigsqcup_{\xi^* \in S^*_f} (0,\infty)\xi^*$$

by Proposition 3.5.39. We have $S^* = S^*_f$. □

Definition 3.5.11

Suppose that S is a basis of a reduced real root system Δ. The subgroup of the Weyl group $W(\Delta)$ generated by the set

$$\{\, \sigma_\xi : \xi \text{ is an element of the basis } S \,\}$$

is denoted by $W(S)$.

Lemma 3.5.1

Suppose that S is a basis of a reduced real root system Δ and let f be a linear functional on the real vector space $\operatorname{span}\Delta$. There exists an element σ of the subgroup $W(S)$ such that we have

$$(f \circ \sigma)(\xi) \geq 0$$

for any element ξ of the basis S.

Proof. There exists an element σ of the subgroup $W(S)$ such that we have

$$(f \circ \sigma)\Big(\sum \Delta^+\Big) \geq (f \circ \tau)\Big(\sum \Delta^+\Big)$$

for any element τ of the subgroup $W(S)$. We have

$$(f \circ \sigma)(\xi) \geq 0$$

for any element ξ of the basis S since we have

$$\begin{aligned} (f \circ \sigma)\Big(\sum \Delta^+\Big) &\geq (f \circ \sigma \circ \sigma_\xi)\Big(\sum \Delta^+\Big) \\ &= (f \circ \sigma)\Big(\sum \Delta^+\Big) - 2(f \circ \sigma)(\xi) \end{aligned}$$

by Proposition 3.5.36. □

Corollary 3.5.3
Suppose that S is a basis of a reduced real root system Δ and let f be a linear functional on the real vector space $\operatorname{span}\Delta$. There exists an element σ of the Weyl group $W(\Delta)$ such that we have

$$(f \circ \sigma)(\xi) \geq 0$$

for any element ξ of the basis S.

Lemma 3.5.2
Suppose that S is a basis of a reduced real root system Δ. We have

$$\begin{aligned}&\big\{\text{ bases of the reduced real root system } \Delta \,\big\} \\ &\qquad\qquad = \big\{\, \sigma(S) : \sigma \text{ is an element of the subgroup } W(S) \,\big\}.\end{aligned}$$

Proof. Suppose that f is a linear functional on the real vector space $\operatorname{span}\Delta$ such that the set $f(\Delta)$ does not contain the element 0. There exists an element σ of the subgroup $W(S)$ such that we have

$$(f \circ \sigma)(\xi) > 0$$

for any element ξ of the basis S. We have $S_f = \sigma(S)$. □

Corollary 3.5.4
The Weyl group of a reduced real root system acts on the set of bases of the reduced real root system transitively.

Lemma 3.5.3
Suppose that S is a basis of a reduced real root system Δ. We have

$$\Delta = W(S)S.$$

Proof. Suppose that ξ is an element of the reduced real root system Δ and let η be an element of the set $\Delta \setminus \big\{\pm\xi\big\}$. The subspace $\ker\xi \cap \ker\eta$ is a proper subset of the subspace $\ker\xi$. There exists a linear functional f_0 on the real vector space $\operatorname{span}\Delta$ such that we have

$$0 = f_0(\xi) < \big|f_0(\eta)\big|$$

for any element η of the set $\Delta \setminus \big\{\pm\xi\big\}$. There exists a linear functional f on the real vector space $\operatorname{span}\Delta$ such that we have

$$0 < f(\xi) < \big|f(\eta)\big|$$

for any element η of the set $\Delta \setminus \big\{\pm\xi\big\}$. The element ξ belongs to the basis S_f and there exists an element σ of the subgroup $W(S)$ such that we have $S_f = \sigma(S)$. □

Corollary 3.5.5
Suppose that S is a basis of a reduced real root system Δ. We have

$$\Delta = W(\Delta)S.$$

Theorem 3.5.9
Suppose that S is a basis of a reduced real root system Δ. We have

$$W(\Delta) = W(S).$$

3.6 Semisimple Lie Algebras

Suppose that g is a finite dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 and let Δ denote the root system. We have the Cartan decomposition

$$g = g_0 \oplus \bigoplus_{\mu \in \Delta} g_\mu.$$

Proposition 3.6.1
Suppose that μ is an element of the space g_0^*. The linear mapping

$$g_\mu \to g_{-\mu}^*, \qquad x \mapsto \big[y \mapsto B(x, y)\big]$$

is an isomorphism and we have $\dim g_\mu = \dim g_{-\mu}$.

Proof. It is sufficient to show that the linear mapping is a monomorphism since the dual mapping of the linear mapping is precisely

$$g_{-\mu} \to g_\mu^*, \qquad y \mapsto \big[x \mapsto B(x, y)\big].$$

Suppose that x is an element of the space g_μ such that we have $B(x, y) = 0$ for any element y of the space $g_{-\mu}$. We have $x \perp g$ by Proposition 3.3.22. We have $x = 0$ since the Killing form is nondegenerate. $\square$

Corollary 3.6.1
We have $\Delta = -\Delta$.

Proposition 3.6.2
Suppose that x and y are elements of the Cartan subalgebra. We have

$$B(x, y) = \sum_{\mu \in \Delta} \mu(x)\mu(y).$$

Proposition 3.6.3
We have $g_0^* = \operatorname{span} \Delta$.

Proof. We have

$$\bigcap_{\mu \in \Delta} \ker \mu = \{0\}. \quad \square$$

Definition 3.6.1
We define an isomorphism

$$g_0 \to, g_0^*, \qquad x \mapsto t^{-1}(x) = \big[y \mapsto B(x,y)\big]$$

and a symmetric form

$$(\mu,\nu) \mapsto B^*(\mu,\nu) = B\big(t(\mu), t(\nu)\big)$$

on the space g_0^*.

Proposition 3.6.4
Suppose that μ is an element of the space g_0^* and let (x,y) be an element of the space $g_\mu \times g_{-\mu}$. We have

$$[x,y] = B(x,y)t(\mu).$$

Proposition 3.6.5
Suppose that μ is an element of the root system Δ. There exists an element (x,y) of the space $g_\mu \times g_{-\mu}$ such that we have

$$B(x,y) = 1, \qquad [x,y] = t(\mu).$$

Proposition 3.6.6
Suppose that μ is an element of the space g_0^* and let (x,y) be an element of the space $g_\mu \times g_{-\mu}$ such that we have $B(x,y) \neq 0$. Suppose that h is an invariant subspace for the derivations $\operatorname{ad} x$ and $\operatorname{ad} y$. We have

$$\operatorname{tr}\big(\operatorname{ad}_h t(\mu)\big) = 0.$$

Proof. We may assume that the element μ is contained in the root system Δ. We have

$$[\operatorname{ad} x, \operatorname{ad} y] = B(x,y) \operatorname{ad} t(\mu).$$

We have

$$0 = \operatorname{tr}[\operatorname{ad}_h x, \operatorname{ad}_h y] = B(x,y) \operatorname{tr}\big(\operatorname{ad}_h t(\mu)\big). \quad \square$$

Proposition 3.6.7
Suppose that μ is a root and let h be a subspace of the Lie algebra g such that the sets $[g_\mu, h]$ and $[g_{-\mu}, h]$ are contained in the space h. We have

$$\operatorname{tr}\big(\operatorname{ad}_h t(\mu)\big) = 0.$$

Proof. We may assume that we have $\mu \neq 0$. There exists an element (x,y) of the space $g_\mu \times g_{-\mu}$ such that we have $B(x,y) \neq 0$. We have

$$\operatorname{tr}\big(\operatorname{ad}_h t(\mu)\big) = 0$$

since the space h is invariant for the derivations $\operatorname{ad} x$ and $\operatorname{ad} y$. $\square$

THEOREM 3.6.1

Suppose that μ and ν are elements of the root system Δ. We have

$$B^*(\mu,\nu) = -\frac{\sum_{n=-\infty}^{\infty} n \dim g_{\mu+n\nu}}{\sum_{n=-\infty}^{\infty} \dim g_{\mu+n\nu}} B^*(\nu,\nu).$$

Proof. We define

$$h = \bigoplus_{n=-\infty}^{\infty} g_{\mu+n\nu}.$$

We have

$$\begin{aligned} 0 = \operatorname{tr}\bigl(\operatorname{ad}_h t(\nu)\bigr) &= \sum_{n=-\infty}^{\infty} (\dim g_{\mu+n\nu}) B^*(\mu+n\nu,\nu) \\ &= (\sum_{n=-\infty}^{\infty} \dim g_{\mu+n\nu}) B^*(\mu,\nu) + (\sum_{n=-\infty}^{\infty} n \dim g_{\mu+n\nu}) B^*(\nu,\nu) \end{aligned}$$

since the sets $[g_\nu, h]$ and $[g_{-\nu}, h]$ are contained in the space h. We have

$$B^*(\mu,\nu) = -\frac{\sum_{n=-\infty}^{\infty} n \dim g_{\mu+n\nu}}{\sum_{n=-\infty}^{\infty} \dim g_{\mu+n\nu}} B^*(\nu,\nu)$$

since we have

$$\sum_{n=-\infty}^{\infty} \dim g_{\mu+n\nu} \geq \dim g_\mu > 0. \quad \square$$

DEFINITION 3.6.2

Suppose that μ and ν are elements of the root system Δ. We define

$$q_{\mu\nu} = -\frac{\sum_{n=-\infty}^{\infty} n \dim g_{\mu+n\nu}}{\sum_{n=-\infty}^{\infty} \dim g_{\mu+n\nu}}.$$

We have

$$B^*(\mu,\nu) = q_{\mu\nu} B^*(\nu,\nu).$$

THEOREM 3.6.2

Suppose that μ is an element of the root system Δ. The element

$$B^*(\mu,\mu) = \frac{1}{\sum_{\nu\in\Delta} q_{\nu\mu}^2}$$

is a positive rational number.

Proof. Suppose that we have $B^*(\mu,\mu) = 0$. We have $B^*(\mu,\nu) = 0$ for any element ν of the root system Δ. We have $\mu = 0$ since we have $g_0^* = \operatorname{span}\Delta$ and the symmetric form B^* is nondegenerate. This is a contradiction. We have $B^*(\mu,\mu) \neq 0$. We have

$$B^*(\mu,\mu) = \frac{1}{\sum_{\nu\in\Delta} (\dim g_\nu) q_{\nu\mu}^2}$$

since we have

$$\begin{aligned} B^*(\mu,\mu) = B\big(t(\mu),t(\mu)\big) &= \sum_{\nu\in\Delta}(\dim g_\nu)\nu\big(t(\mu)\big)^2 \\ &= \sum_{\nu\in\Delta}(\dim g_\nu)B^*(\mu,\nu)^2 \\ &= \sum_{\nu\in\Delta}(\dim g_\nu)q_{\nu\mu}^2 B^*(\mu,\mu)^2. \end{aligned}$$

□

THEOREM 3.6.3
Suppose that μ is an element of the root system Δ.

1. We have $\dim g_\mu = 1$.
2. We have $\mathbb{Z}\mu\cap\Delta = \{\pm\mu\}$.

Proof. There exists an element (x,y) of the space $g_\mu\times g_{-\mu}$ such that we have $B(x,y)\neq 0$. We define

$$h = \mathbb{F}y\oplus\mathbb{F}t(\mu)\oplus\bigoplus_{n=1}^{\infty} g_{n\mu}.$$

The space h is invariant for the derivations $\operatorname{ad} x$ and $\operatorname{ad} y$ since we have

$$\begin{aligned} (\operatorname{ad} y)\big(t(\mu)\big) &= -\big(\operatorname{ad} t(\mu)\big)(y) \\ &= \mu\big(t(\mu)\big)y = B^*(\mu,\mu)y. \end{aligned}$$

We have

$$0 = \operatorname{tr}\big(\operatorname{ad}_h t(\mu)\big) = B^*(\mu,\mu)(-1+\sum_{n=1}^{\infty} n\dim g_{n\mu}).$$

We have $\dim g_\mu = 1$ and we have $\dim g_{n\mu} = 0$ for any $n > 1$. □

THEOREM 3.6.4
Suppose that (μ,ν) is an element of the set $(\Delta\cup\{0\})\times\Delta$ and let

$$\begin{aligned} n_- &= \max\{\, n\in\mathbb{Z}_+ : \mu - n\nu \text{ is a root}\,\}, \\ n_+ &= \max\{\, n\in\mathbb{Z}_+ : \mu + n\nu \text{ is a root}\,\}. \end{aligned}$$

The set $\{\mu+n\nu\}_{n=-n_-}^{n_+}$ is contained in the set $\Delta\cup\{0\}$ and we have

$$n_- - n_+ = \frac{2B^*(\mu,\nu)}{B^*(\nu,\nu)}.$$

Proof. We define

$$\begin{aligned} n_- &= \min\{\, n\in\mathbb{N} : \mu - n\nu \text{ is not a root}\,\} - 1 \geq 0, \\ n_+ &= \min\{\, n\in\mathbb{N} : \mu + n\nu \text{ is not a root}\,\} - 1 \geq 0. \end{aligned}$$

We define

$$h = \bigoplus_{n=-n_-}^{n_+} g_{\mu+n\nu}.$$

We have

$$\begin{aligned} 0 = \operatorname{tr}\big(\operatorname{ad}_h t(\nu)\big) &= \sum_{n=-n_-}^{n_+} (\mu + n\nu)\big(t(\nu)\big) \\ &= (n_+ + n_- + 1)\Big(B^*(\mu,\nu) + \frac{(n_+ - n_-)}{2} B^*(\nu,\nu)\Big). \end{aligned}$$

since the sets $[g_\nu, h]$ and $[g_{-\nu}, h]$ are contained in the space h. We have

$$n_- - n_+ = \frac{2B^*(\mu,\nu)}{B^*(\nu,\nu)}.$$

Suppose that there exists an integer $-n_- < n < n_+$ such that the element $\mu + n\nu$ is not contained in the set $\Delta \cup \{0\}$. We have

$$\begin{aligned} 0 &\le \frac{2B^*\big(\mu + (n-1)\nu, \nu\big)}{B^*(\nu,\nu)} = \frac{2B^*(\mu,\nu)}{B^*(\nu,\nu)} + 2(n-1) \\ &< \frac{2B^*\big(\mu + (n+1)\nu, \nu\big)}{B^*(\nu,\nu)} = \frac{2B^*(\mu,\nu)}{B^*(\nu,\nu)} + 2(n+1) \le 0. \end{aligned}$$

This is a contradiction. □

Definition 3.6.3
Suppose that (μ,ν) is an element of the set $(\Delta \cup \{0\}) \times \Delta$. We define

$$c_{\mu\nu} = \frac{2B^*(\mu,\nu)}{B^*(\nu,\nu)}.$$

Remark 3.6.1
The element $\mu - c_{\mu\nu}\nu$ is contained in the set $\Delta \cup \{0\}$.

Proof. We have $-n_- \le -c_{\mu\nu} = n_+ - n_- \le n_+$. □

Proposition 3.6.8
[3] Suppose that μ is an element of the root system Δ. We have

$$\mu^* = \frac{2t(\mu)}{B^*(\mu,\mu)}.$$

Theorem 3.6.5
The root system of a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0 is reduced.

[3]Proposition 3.5.9.

THEOREM 3.6.6

Suppose that μ is an element of the root system Δ. The subspace

$$g_\mu \oplus g_{-\mu} \oplus \mathbb{F}\mu^*$$

is a Lie subalgebra isomorphic to the Lie algebra $M(2, \mathbb{F}) \cap \ker \operatorname{tr}$.

Proof. There exists an element (x, y) of the space $g_\mu \times g_{-\mu}$ such that we have $[x, y] = \mu^*$. We have $[\mu^*, x] = 2x$ and we have $[\mu^*, y] = -2y$. □

THEOREM 3.6.7

Suppose that μ and ν are elements of the root system Δ such that the element $\mu + \nu$ is contained in the root system Δ and let

$$n_- = \max\{\, n \in \mathbb{Z}_+ : \mu - n\nu \text{ is a root} \,\},$$
$$n_+ = \max\{\, n \in \mathbb{Z}_+ : \mu + n\nu \text{ is a root} \,\}.$$

The adjoint representation of the Lie algebra

$$M(2, \mathbb{F}) \cap \ker \operatorname{tr} = g_\nu \oplus g_{-\nu} \oplus \mathbb{F}\nu^*$$

on the vector space

$$\bigoplus_{n=-n_-}^{n_+} g_{\mu+n\nu}$$

is irreducible.

Proof. The representation is completely reducible by Weyl's theorem. The set $\{\mu + n\nu\}_{n=-n_-}^{n_+}$ is contained in the root system Δ and we have

$$(\mu + n_+\nu)(\nu^*) = n_+ + n_-. \quad \square$$

COROLLARY 3.6.2

Suppose that μ and ν are elements of the root system Δ such that the element $\mu + \nu$ is contained in the root system Δ. We have $[g_\mu, g_\nu] = g_{\mu+\nu}$.

PROPOSITION 3.6.9

The set $\Delta(\operatorname{span}_\mathbb{Q} \Delta^*)$ is contained in the field $\mathbb{Q}$.

PROPOSITION 3.6.10

We have $\operatorname{rank} g = \dim \operatorname{span}_\mathbb{Q} \Delta = \dim_\mathbb{Q} \operatorname{span}_\mathbb{Q} \Delta^*$.

PROPOSITION 3.6.11

We have $g_0^* = \operatorname{span}_\mathbb{Q} \Delta \otimes \mathbb{F}$.

Proof. There exists a canonical epimorphism of the space $\operatorname{span}_\mathbb{Q} \Delta \otimes \mathbb{F}$ onto the space g_0^*. We have $\dim g_0^* = \dim(\operatorname{span}_\mathbb{Q} \Delta \otimes \mathbb{F}) = \operatorname{rank} g$. □

PROPOSITION 3.6.12

We have $(\operatorname{span}_\mathbb{Q} \Delta)^* = \operatorname{span}_\mathbb{Q} \Delta^*$.

Proposition 3.6.13

The Killing form induces the inner product on the real vector space $(\operatorname{span}_{\mathbb{Q}} \Delta \otimes \mathbb{R})^* = \operatorname{span}_{\mathbb{Q}} \Delta^* \otimes \mathbb{R}$.

Proof. Suppose that S is a basis of the space g_0^* that is contained in the root system Δ. The symmetric matrix

$$\left(B(\mu^*, \nu^*) = \frac{4B^*(\mu, \nu)}{B^*(\mu, \mu)B^*(\nu, \nu)} \right)_{\mu, \nu \in S}$$

belongs to the group $M(\operatorname{rank} g, \mathbb{Q})^\times$ and we have

$$B(x, x) = \sum_{\mu \in \Delta} \mu(x)^2 \geq 0$$

for any element x of the space $\operatorname{span}_{\mathbb{Q}} \Delta^*$. The space $\operatorname{span}_{\mathbb{Q}} \Delta^*$ is a dense subspace of the space $\operatorname{span}_{\mathbb{Q}} \Delta^* \otimes \mathbb{R}$ by Proposition 2.2.11. Suppose that x is an arbitrary element of the space $\operatorname{span}_{\mathbb{Q}} \Delta^* \otimes \mathbb{R}$. There exists a sequence $(x_n)_{n=1}^{\infty}$ of the space $\operatorname{span}_{\mathbb{Q}} \Delta^*$ such that we have $x = \lim_{n \to \infty} x_n$. We have

$$\begin{aligned} B(x, x) &= \lim_{n \to \infty} B(x_n, x_n) \\ &= \lim_{n \to \infty} \sum_{\mu \in \Delta} \mu(x_n)^2 = \sum_{\mu \in \Delta} \mu(x)^2 \geq 0. \end{aligned}$$

The nonnegative symmetric form on the real vector space $\operatorname{span}_{\mathbb{Q}} \Delta^* \otimes \mathbb{R}$ is an inner product since it is nondegenerate. □

Suppose that $\Delta^+ \setminus (\Delta^+ + \Delta^+)$ is a basis of the reduced real root system Δ and let

$$g^+ = \bigoplus_{\mu \in \Delta^+} g_\mu, \qquad g^- = \bigoplus_{\mu \in \Delta^+} g_{-\mu}.$$

Proposition 3.6.14

Any element of the Lie subalgebras $g^\pm$ is nilpotent.

Corollary 3.6.3

The Lie algebras $g^\pm$ are nilpotent.

Corollary 3.6.4

The Lie algebras $g_0 \oplus g^\pm$ are solvable and we have $D(g_0 \oplus g^\pm) = g^\pm$.

Suppose that we have $\Delta^+ \setminus (\Delta^+ + \Delta^+) = \{\mu_k\}_{k=1}^{\operatorname{rank} g}$ and let (x_k, y_k) be an element of the space $g_{\mu_k} \times g_{-\mu_k}$ such that we have $[x_k, y_k] = \mu_k^*$ for any k.

Theorem 3.6.8

We have the following.

1. The Lie algebra g^+ is generated by the set $\{x_k\}_{k=1}^{\operatorname{rank} g}$.

2. The Lie algebra g^- is generated by the set $\{y_k\}_{k=1}^{\operatorname{rank} g}$.

3. The Lie algebra g is generated by the set $\{x_k\}_{k=1}^{\operatorname{rank} g} \cup \{y_k\}_{k=1}^{\operatorname{rank} g}$.

THEOREM 3.6.9
We have the following.

1. We have $[\mu_i^*, \mu_j^*] = 0$.

2. We have $[x_i, y_j] = \delta_{ij}\mu_j^*$.

3. We have $[\mu_i^*, x_j] = C_{\mu_j\mu_i} x_j$.

4. We have $[\mu_i^*, y_j] = -C_{\mu_j\mu_i} y_j$.

5. We have
$$(\operatorname{ad} x_i)^{-C_{\mu_j\mu_i}+1}(x_j) = (\operatorname{ad} y_i)^{-C_{\mu_j\mu_i}+1}(y_j) = 0$$
provided that the elements μ_i and μ_j are distinct.

Proof. 1. The Cartan subalgebra is abelian.

2. Suppose that the elements μ_i and μ_j are distinct. The elements $[x_i, y_j]$ belongs to the space $g_{\mu_i-\mu_j} = \{0\}$.

3. We have $[\mu_i^*, x_j] = \mu_j(\mu_i^*)x_j = C_{\mu_j\mu_i} x_j$.

4. We have $[\mu_i^*, y_j] = -\mu_j(\mu_i^*)y_j = -C_{\mu_j\mu_i} y_j$.

5. We have
$$-C_{\mu_j\mu_i} = \max\{\, n \in \mathbb{Z}_+ : \mu_j + n\mu_i \text{ is a root} \,\}.$$
The element $(\operatorname{ad} x_i)^{-C_{\mu_j\mu_i}+1}(x_j)$ belongs to the space
$$g_{\mu_j+(-C_{\mu_j\mu_i}+1)\mu_i} = \{0\}. \quad \square$$

REMARK 3.6.2
We have
$$\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1} \end{pmatrix}\right)^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix}.$$

EXAMPLE 3.6.1
Suppose that m is a positive integer.
$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \operatorname{diag}(x_1, \ldots, x_m) \\ 0 & -\operatorname{diag}(x_1, \ldots, x_m) & 0 \end{pmatrix} : x_1, \ldots, x_m \in \mathbb{F} \right\}$$
is a Cartan subalgebra of the semisimple Lie algebra
$$\{\, x \in M(2m+1, \mathbb{F}) : x^T = -x \,\}.$$

We have

$$\begin{aligned}&\left(1 \oplus \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \sqrt{-1}\\ 1 & -\sqrt{-1}\end{pmatrix}\right)\\ &\qquad \left\{\, x \in M(2m+1,\mathbb{F}) : x^T = -x \,\right\}\left(1 \oplus \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -\sqrt{-1} & \sqrt{-1}\end{pmatrix}\right)\\ &\qquad\qquad = \left\{ \begin{pmatrix}0 & x_{12} & x_{13}\\ -x_{13}^T & x_{22} & x_{23}\\ -x_{12}^T & x_{32} & -x_{22}^T\end{pmatrix} : x_{23} + x_{23}^T = x_{32} + x_{32}^T = 0 \right\}.\end{aligned}$$

We have

$$\begin{aligned}&\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \sqrt{-1}\\ 1 & -\sqrt{-1}\end{pmatrix}\begin{pmatrix}0 & \sqrt{-1}\operatorname{diag}(x_1,\dots,x_m)\\ -\sqrt{-1}\operatorname{diag}(x_1,\dots,x_m) & 0\end{pmatrix}\\ &\qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -\sqrt{-1} & \sqrt{-1}\end{pmatrix} = \begin{pmatrix}\operatorname{diag}(x_1,\dots,x_m) & 0\\ 0 & -\operatorname{diag}(x_1,\dots,x_m)\end{pmatrix}.\end{aligned}$$

Chapter 4

Universal Enveloping Algebras

4.1 Poincaré-Birkhoff-Witt Theorem

Suppose that g is a Lie algebra over a field $\mathbb{F}$ and let X be its basis. By the well-ordering theorem there exists a total order on the set X. We write π for the canonical homomorphism of the tensor algebra $T(g)$ onto the universal enveloping algebra $U(g)$. We define

$$U_n(g) = \pi\big(\bigoplus_{k=0}^{n} T^k(g)\big)$$

for any nonnegative integer n. We have $\lim_{n\to\infty} U_n(g) = U(g)$.

Remark 4.1.1

Suppose that n is a nonnegative integer and let f be a linear mapping of the vector space $\bigoplus_{k=0}^{n} T^k(g)$ into any vector space such that we have

$$f(x_1 \cdots x_m) = f(x_1 \cdots x_k x_{k-1} \cdots x_m) + f\big(x_1 \cdots [x_{k-1}, x_k] \cdots x_m\big)$$

for any element x of the set

$$\{\, x \in X^m : x_k < x_{k-1} \,\}$$

for any integers m and k. We have

$$f\Big(x_1 \cdots \big(x_{k-1}x_k - x_k x_{k-1} - [x_{k-1}, x_k]\big) \cdots x_m\Big) = 0$$

for any element x of the set g^m for any integers m and k.

Proposition 4.1.1

The universal enveloping algebra $U(g)$ is linearly generated by the set

$$\bigcup_{n=0}^{\infty} \{\, \pi(x_1) \cdots \pi(x_n) : x_1 \le \cdots \le x_n \in X \,\}.$$

Proof. It is sufficient to show that the subspace $U_n(g)$ is linearly generated by the set

$$\bigcup_{m=0}^{n} \{\, \pi(x_1)\cdots\pi(x_m) : x_1 \leq \cdots \leq x_m \in X \,\} \tag{4.1}$$

for any nonnegative integer n. The proof is by induction on the nonnegative integer n. Suppose that we have $n > 0$. It is sufficient to show that the element $\pi(x_1)\cdots\pi(x_n)$ is contained in the subspace generated by the set (4.1) for any element x of the set X^n. The proof is by induction on the nonnegative integer

$$r(x) = \#\{\, (i,j) : i < j \text{ and } x_j < x_i \,\}.$$

Suppose that we have $r(x) > 0$. We have $x_k < x_{k-1}$ for some integer k. The element

$$\begin{aligned}\pi(x_1)\cdots\pi(x_n) = \pi(x_1)\cdots\pi(x_k)\pi(x_{k-1})\cdots\pi(x_n)& \\ + \pi(x_1)\cdots\pi\big([x_{k-1}, x_k]\big)\cdots\pi(x_n)&\end{aligned}$$

is contained in the subspace generated by the set (4.1) by the induction hypothesis since we have $r(x_1, \ldots, x_k, x_{k-1}, \ldots, x_n) = r(x) - 1$ and the element

$$\pi(x_1)\cdots\pi\big([x_{k-1}, x_k]\big)\cdots\pi(x_n)$$

belongs to the subspace $U_{n-1}(g)$. □

PROPOSITION 4.1.2

There exists a unique linear mapping f of the tensor algebra $T(g)$ onto the symmetric algebra $S(g)$ satisfying the following.

1. We have
$$f(x_1\cdots x_n) = x_1\cdots x_n$$
for any element x of the set
$$\{\, x \in X^n : x_1 \leq \cdots \leq x_n \,\}$$
for any nonnegative integer n.

2. We have
$$f(x_1\cdots x_n) = f(x_1\cdots x_k x_{k-1}\cdots x_n) + f\big(x_1\cdots [x_{k-1}, x_k]\cdots x_n\big)$$
for any element x of the set
$$\{\, x \in X^n : x_k < x_{k-1} \,\}$$
for any integers n and k.

Proof. It is sufficient to show the following proposition. □

Proposition 4.1.3

Suppose that f is a linear mapping of the subspace

$$\bigoplus_{k=0}^{n-1} T^k(g) \oplus \operatorname{span}\Big\{\, x_1 \cdots x_n : x \text{ is an element of the set } X^n \\ \text{such that we have } r(x) < r \,\Big\}$$

into any vector space such that we have

$$f(x_1 \cdots x_m) = f(x_1 \cdots x_k x_{k-1} \cdots x_m) + f\big(x_1 \cdots [x_{k-1}, x_k] \cdots x_m\big)$$

for any element x of the set

$$\big\{\, x \in X^m : x_k < x_{k-1} \,\big\}$$

for any integers m and k. We have

$$f(x_1 \cdots x_{k_1} x_{k_1-1} \cdots x_n) + f\big(x_1 \cdots [x_{k_1-1}, x_{k_1}] \cdots x_n\big) \\ = f(x_1 \cdots x_{k_2} x_{k_2-1} \cdots x_n) + f\big(x_1 \cdots [x_{k_2-1}, x_{k_2}] \cdots x_n\big)$$

for any element x of the set

$$\big\{\, x \in X^n : x_{k_1} < x_{k_1-1} \text{ and } x_{k_2} < x_{k_2-1} \,\big\} \cap r^{-1}(r)$$

for any integers k_1 and k_2.

Proof. Suppose that we have $k = k_1 = k_2 - 1$. We have

$$f\Big(\cdots \big(x_k x_{k-1} + [x_{k-1}, x_k]\big) \cdots\Big) - f\big(\cdots x_{k+1} x_k x_{k-1} \cdots\big) \\ = f\Big(\cdots \big([x_k, x_{k+1}]x_{k-1} + x_k[x_{k-1}, x_{k+1}] + [x_{k-1}, x_k]x_{k+1}\big) \cdots\Big)$$

and we have

$$f\Big(\cdots \big(x_{k+1} x_k + [x_k, x_{k+1}]\big) \cdots\Big) - f\big(\cdots x_{k+1} x_k x_{k-1} \cdots\big) \\ = f\Big(\cdots \big(x_{k+1}[x_{k-1}, x_k] + [x_{k-1}, x_{k+1}]x_k + x_{k-1}[x_k, x_{k+1}]\big) \cdots\Big).$$

We have

$$f\Big(\cdots \big(x_k x_{k-1} + [x_{k-1}, x_k]\big) \cdots\Big) - f\Big(\cdots \big(x_{k+1} x_k + [x_k, x_{k+1}]\big) \cdots\Big) \\ = f\Big(\cdots \Big(\big[[x_k, x_{k+1}], x_{k-1}\big] + \big[x_k, [x_{k-1}, x_{k+1}]\big] + \big[[x_{k-1}, x_k], x_{k+1}\big]\Big) \cdots\Big).$$

Suppose that we have $k_1 < k_2 - 1$. We have

$$f\Big(\cdots \big(x_{k_1} x_{k_1-1} + [x_{k_1-1}, x_{k_1}]\big) \cdots\Big) = f\Big(\cdots \big(x_{k_2} x_{k_2-1} + [x_{k_2-1}, x_{k_2}]\big) \cdots\Big) \\ = f\Big(\cdots \big(x_{k_1} x_{k_1-1} + [x_{k_1-1}, x_{k_1}]\big) \cdots \big(x_{k_2} x_{k_2-1} + [x_{k_2-1}, x_{k_2}]\big) \cdots\Big). \quad \square$$

Corollary 4.1.1

There exists a unique linear mapping f of the universal enveloping algebra $U(g)$ onto the symmetric algebra $S(g)$ such that we have

$$f\big(\pi(x_1)\cdots\pi(x_n)\big) = x_1\cdots x_n$$

for any element x of the set

$$\left\{\, x \in X^n : x_1 \leq \cdots \leq x_n \,\right\}$$

for any nonnegative integer n.

Theorem 4.1.1 (Poincaré-Birkhoff-Witt)

We have the following.

1. The canonical homomorphism of the Lie algebra g into the universal enveloping algebra $U(g)$ is an imbedding.
2. The linear mapping f of the symmetric algebra $S(g)$ onto the universal enveloping algebra $U(g)$ defined by

   $$f(x_1\cdots x_n) = x_1\cdots x_n$$

   for any nonnegative integer n and for any element x of the set

   $$\left\{\, x \in X^n : x_1 \leq \cdots \leq x_n \,\right\}$$

   is a linear isomorphism.

Proof. The linear mapping f of the symmetric algebra $S(g)$ onto the universal enveloping algebra $U(g)$ defined by

$$f(x_1\cdots x_n) = \pi(x_1)\cdots\pi(x_n)$$

for any nonnegative integer n and for any element x of the set

$$\left\{\, x \in X^n : x_1 \leq \cdots \leq x_n \,\right\}$$

is an imbedding by the previous proposition. □

Proposition 4.1.4

The universal enveloping algebra of a Lie subalgebra is a subalgebra with identity of the universal enveloping algebra of the Lie algebra.

Definition 4.1.1

We define $U_{-1}(g) = \{0\}$ and we define

$$\operatorname{gr}_n U(g) = \frac{U_n(g)}{U_{n-1}(g)}$$

for any nonnegative integer n.

Proposition 4.1.5
We have

$$\operatorname{gr}_0 U(g) = U_0(g) = \mathbb{F}, \qquad \operatorname{gr}_1 U(g) = \frac{U_1(g)}{U_0(g)} = \frac{\mathbb{F} \oplus g}{\mathbb{F}} = g.$$

Proposition 4.1.6
The mapping

$$\begin{aligned} &\operatorname{gr}_m U(g) \times \operatorname{gr}_n U(g) \to \operatorname{gr}_{m+n} U(g), \\ &\qquad \big(x + U_{m-1}(g), y + U_{n-1}(g)\big) \mapsto xy + U_{m+n-1}(g) \end{aligned}$$

is bilinear for any nonnegative integers m and n and the direct sum

$$\operatorname{gr} U(g) = \bigoplus_{n=0}^{\infty} \operatorname{gr}_n U(g)$$

is a graded commutative algebra with identity.

Theorem 4.1.2
We have $S(g) = \operatorname{gr} U(g)$.

Suppose that g_1 and g_2 are Lie algebras over a field.

Proposition 4.1.7
A linear mapping f of the algebra $U(g_1)$ into the algebra $U(g_2)$ such that the space $f\big(U_n(g_1)\big)$ is contained in the space $U_n(g_2)$ for any nonnegative integer n induces a linear mapping $\operatorname{gr} f$ of the algebra $S(g_1) = \operatorname{gr} U(g_1)$ into the algebra $S(g_2) = \operatorname{gr} U(g_2)$ such that we have

$$\operatorname{gr} f\big(x_n + U_{n-1}(g_1)\big) = f(x_n) + U_{n-1}(g_2)$$

for any element x_n of the space $U_n(g_1)$ for any nonnegative integer n.

1. We have $\operatorname{gr} f(1) = 1$ provided that we have $f(1) = 1$.
2. The mapping $\operatorname{gr} f$ is a homomorphism of graded algebras provided that the mapping f is a homomorphism of algebras.

Proof. Suppose that the linear mapping f is a homomorphism of algebras. We have

$$\operatorname{gr} f(xy) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f(x_k) g(y_{n-k}) + U_{n-1}(g_2) = \operatorname{gr} f(x) \operatorname{gr} f(y)$$

for any elements x and y of the algebra $\operatorname{gr} U(g_1)$. □

Theorem 4.1.3
We have $U(g_1 \oplus g_2) = U(g_1) \otimes U(g_2)$.

Suppose that V_1 and V_2 are vector spaces over a field.

Corollary 4.1.2
We have $S(V_1 \oplus V_2) = S(V_1) \otimes S(V_2)$.

Proof. Any vector space is an abelian Lie algebra. □

Proposition 4.1.8
The inclusion mapping of the algebra $U(g_1)$ into the algebra $U(g_1) \otimes U(g_2)$ induces the inclusion mapping of the algebra $S(g_1) = \operatorname{gr} U(g_1)$ into the algebra $S(g_1) \otimes S(g_2) = \operatorname{gr}\bigl(U(g_1) \otimes U(g_2)\bigr)$.

Proposition 4.1.9
We have
$$x \otimes 1 + U_{n-1}(g_1 \oplus g_2) = \bigl(x + U_{n-1}(g_1)\bigr) \otimes 1$$
for any element x of the space $U_n(g_1)$ for any nonnegative integer n.

Proof. We write f for the inclusion mapping of the algebra $U(g_1)$ into the algebra $U(g_1) \otimes U(g_2)$. We have
$$x \otimes 1 + U_{n-1}(g_1 \oplus g_2) = \operatorname{gr} f\bigl(x + U_{n-1}(g_1)\bigr) = \bigl(x + U_{n-1}(g_1)\bigr) \otimes 1. \quad \square$$

Proposition 4.1.10
The diagonal mapping Δ is the unique homomorphism of algebras with identity of the algebra $U(g)$ into the algebra $U(g) \otimes U(g)$ such that we have
$$\Delta(x) = x \otimes 1 + 1 \otimes x$$
for any element x of the Lie algebra g.

Proposition 4.1.11
The induced mapping $\operatorname{gr} \Delta$ is a homomorphism of graded algebras with identity of the algebra $S(g)$ into the algebra $S(g) \otimes S(g)$ such that we have
$$\operatorname{gr} \Delta(x) = x \otimes 1 + 1 \otimes x$$
for any element x of the Lie algebra g.

Definition 4.1.2
An element x of the universal enveloping algebra $U(g)$ is said to be primitive if we have $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Theorem 4.1.4
Suppose that the field $\mathbb{F}$ is of characteristic 0. Any primitive element of the universal enveloping algebra $U(g)$ belongs to the Lie algebra g.

Proof. Suppose that the Lie algebra g is abelian and let $\boldsymbol{x}$ be its basis. We have
$$U(g) = \mathbb{F}[\boldsymbol{x}], \qquad U(g) \otimes U(g) = U(g \oplus g) = \mathbb{F}[\boldsymbol{x}_1, \boldsymbol{x}_2].$$

We have

$$\Delta f(\boldsymbol{x}) = f(\boldsymbol{x}_1 + \boldsymbol{x}_2)$$

for any element $f(\boldsymbol{x})$ of the universal enveloping algebra $U(g) = \mathbb{F}[\boldsymbol{x}]$ since we have

$$\Delta(x) = x \otimes 1 + 1 \otimes x = x_1 + x_2$$

for any element x of the basis $\boldsymbol{x}$. Suppose that $f(\boldsymbol{x})$ is a primitive element of the universal enveloping algebra $U(g) = \mathbb{F}[\boldsymbol{x}]$. We have

$$\sum_{n=0}^{\infty} 2^n f_n(\boldsymbol{x}) = f(2\boldsymbol{x}) = 2f(\boldsymbol{x}) = \sum_{n=0}^{\infty} 2f_n(\boldsymbol{x})$$

since we have

$$f(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \Delta f(\boldsymbol{x}) = f(\boldsymbol{x}) \otimes 1 + 1 \otimes f(\boldsymbol{x}) = f(\boldsymbol{x}_1) + f(\boldsymbol{x}_2).$$

The element $f(\boldsymbol{x}) = f_1(\boldsymbol{x})$ belongs to the Lie algebra g.

Suppose that the Lie algebra g is arbitrary and let x be a primitive element of the universal enveloping algebra $U(g)$. We define

$$n = \min\bigl\{\, n \in \mathbb{N} : x \in U_n(g) \,\bigr\}.$$

The element $x + U_{n-1}(g)$ belongs to the space $\operatorname{gr}_1 U(g) = g$ since we have

$$\begin{aligned}
\operatorname{gr} \Delta\bigl(x + U_{n-1}(g)\bigr) &= x \otimes 1 + 1 \otimes x + U_{n-1}(g \oplus g) \\
&= \bigl(x \otimes 1 + U_{n-1}(g \oplus g)\bigr) + \bigl(1 \otimes x + U_{n-1}(g \oplus g)\bigr) \\
&= \bigl(x + U_{n-1}(g)\bigr) \otimes 1 + 1 \otimes \bigl(x + U_{n-1}(g)\bigr)
\end{aligned}$$

by Proposition 4.1.9. We have $n = 1$ since the element $x + U_{n-1}(g)$ is primitive. There exists a scalar ν such that the element $x - \nu$ belongs to the Lie algabra g. We have $\nu = x \otimes 1 + 1 \otimes x - \Delta(x) = 0$. □

4.2 Free Lie Algebras

In this section algebras are not necessarily associative.

Definition 4.2.1
A set with a binary operation is called a magma.

Proposition 4.2.1
A set is a subset of the free magma on the set.

Suppose that X is a set. We write M_X for the free magma on the set X.

Theorem 4.2.1
Suppose that f is a mapping of the set X into a magma M. There exists a unique homomorphism of the magma M_X into the magma M extending the mapping f.

Suppose that $\mathbb{F}$ is a field.

Proposition 4.2.2
There exists a unique bilinear binary operation on the vector space $\mathbb{F}^{\oplus M_X}$ extending the binary operation on the magma M_X.

Proposition 4.2.3
The vector space $\mathbb{F}^{\oplus M_X}$ is an algebra over the field $\mathbb{F}$.

Proposition 4.2.4
The magma M_X is a submagma of the algebra $\mathbb{F}^{\oplus M_X}$.

Proposition 4.2.5
The algebra $\mathbb{F}^{\oplus M_X} = \bigoplus_{n=1}^{\infty} \mathbb{F}^{\oplus X_n}$ is graded.

Theorem 4.2.2
Suppose that f is a mapping of the set X into an algebra A. There exists a unique homomorphism of the algebra $\mathbb{F}^{\oplus M_X}$ into the algebra A extending the mapping f.

We write L_X for the free Lie algebra on the set X.

Theorem 4.2.3
We have $U(L_X) = \mathbb{F}\langle X\rangle = T(\mathbb{F}^{\oplus X})$.

Proposition 4.2.6
The Lie algebra L_X is generated by the set X.

Theorem 4.2.4
Suppose that f is a mapping of the set X into a Lie algebra L. There exists a unique homomorphism of the Lie algebra L_X into the Lie algebra L extending the mapping f.

Definition 4.2.2
We define

$$L_X^n = L_X \cap \mathbb{F}\langle X\rangle^n, \qquad \mathbb{F}\langle X\rangle^n = \operatorname{span}\{\, x_1 \cdots x_n : x_1, \ldots, x_n \in X \,\}$$

for any nonnegative integer n.

Proposition 4.2.7
The Lie algebra $L_X = \bigoplus_{n=1}^{\infty} L_X^n$ is graded.

Proposition 4.2.8
We have

$$L_X^n = \operatorname{span}\{\, (\operatorname{ad} x_1) \cdots (\operatorname{ad} x_{n-1})(x_n) : (x_k)_{k=1}^n \in X^n \,\}$$

for any positive integer n.

Proposition 4.2.9
We have the following.

1. The kernel of the canonical homomorphism of the graded algebra $\mathbb{F}^{\oplus M_X}$ onto the graded Lie algebra L_X is the ideal generated by the following elements.

$$\begin{aligned}
[x, x], &\qquad x \in \mathbb{F}^{\oplus M_X} \\
[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2], &\qquad x_1, x_2, x_3 \in \mathbb{F}^{\oplus M_X}
\end{aligned}$$

2. The kernel of the canonical homomorphism of the graded algebra $\mathbb{F}^{\oplus M_X}$ onto the graded Lie algebra L_X is the ideal generated by the following homogeneous elements.

$$\begin{aligned}
[x, x], &\qquad x \in M_X \\
[x_1, x_2] + [x_2, x_1], &\qquad x_1 \neq x_2 \in M_X \\
[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2], &\qquad x_1, x_2, x_3 \in M_X
\end{aligned}$$

Proposition 4.2.10
The associative algebra $\mathbb{F}\langle X\rangle = \bigoplus_{n=0}^{\infty} \mathbb{F}\langle X\rangle^n$ is graded and its completion $\overline{\mathbb{F}\langle X\rangle} = \prod_{n=0}^{\infty} \mathbb{F}\langle X\rangle^n$ is an associative algebra.

Proposition 4.2.11
The completion $\overline{L_X} = \prod_{n=1}^{\infty} L_X^n$ is a Lie subalgebra of the ideal $\prod_{n=1}^{\infty} \mathbb{F}\langle X\rangle^n$.

Proof. We have

$$\begin{aligned}
[f, g] &= \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} [f_k, g_{n-k}] \\
&= \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (f_k g_{n-k} - g_{n-k} f_k) = fg - gf
\end{aligned}$$

for any elements f and g of the completion $\overline{L_X}$. □

Proposition 4.2.12
We have the following.

1. The diagonal mapping of the universal enveloping algebra $\mathbb{F}\langle X\rangle$ into the universal enveloping algebra $\mathbb{F}\langle X\rangle \otimes \mathbb{F}\langle X\rangle$ is graded.

2. The diagonal mapping extends to the homomorphism of algebras with identity
$$f = \sum_{n=0}^{\infty} f_n \mapsto \Delta(f) = \sum_{n=0}^{\infty} \Delta(f_n)$$
of the completion $\overline{\mathbb{F}\langle X\rangle}$ into the completion $\overline{\mathbb{F}\langle X\rangle \otimes \mathbb{F}\langle X\rangle}$.

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