Quantum Derivation and Mishchenko-Fomenko Construction

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Outline

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- I will explain and develop a quantum analogue of the theorem of A. Mishchenko and A. Fomenko.
- Main topics of our research:
  1. Formula for Quantum Argument Shift:

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1}\left(f_+^{(n-m-1)}(e)_j(e^m)^i + f_-^{(n-m-1)}(e)(e^m)^i_j\right)$$

  2. Generators of Second Quantum Argument Shift:

$$C^{(2)}_\xi = C^{(1)}_\xi\left[\tau_\xi\left(P_n^{(n)}\right), \tau_\xi\left(P_{n+1}^{(n)} + P_n^{(n+1)}\right) : n \in \mathbb{N}\right]$$

  3. Quantum M-F Theorem (G. Sharygin will talk):

$$\left[\partial_\xi^m(x), \partial_\xi^n(y)\right] = 0$$

Poisson Bracket on the Dual Space of a Lie Algebra

- The Lie bracket of a Lie algebra $\mathfrak{g}$ extends uniquely to a Poisson bracket on the symmetric algebra $S(\mathfrak{g})$. This Poisson bracket is called the Lie-Poisson bracket.

$$\begin{array}{ccc} S(\mathfrak{g}) \times S(\mathfrak{g}) & \xrightarrow{\text{Lie-Poisson}} & S(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\text{Lie}} & \mathfrak{g} \end{array}$$

- The universal enveloping algebra $U(\mathfrak{g})$ is identified with the deformation quantisation of the Poisson algebra $S(\mathfrak{g})$ under the condition $\hbar = 1$.

Theorem of A. Mishchenko and A. Fomenko

Theorem (A. Mishchenko and A. Fomenko, 1978)

Suppose that ∂_ξ is a constant vector field on the dual space $\mathfrak{g}^*$. We have

$$\{\partial_\xi^m(x), \partial_\xi^n(y)\} = 0$$

for any m and n and for any Poisson central elements x and y of the symmetric algebra $S(\mathfrak{g})$.

Argument Shift Algebra and Vinberg's Problem

- The Poisson commutative algebra $\overline{C_\xi}$ generated by

$$\left\{ x, \partial_\xi(x), \partial_\xi^2(x), \cdots : x \text{ is Poisson central} \right\}$$

  is called the argument shift algebra.
- Vinberg asked whether the algebra $\overline{C_\xi}$ can be quantised to the commutative subalgebra C_ξ of the universal enveloping algebra with $\operatorname{gr} C_\xi = \overline{C_\xi}$.
- Vinberg's problem is solved by the Feigin-Frenkel center
  - for regular elements ξ (Feigin et al. and Rybnikov).
  - for simple Lie algebras of types A and C (Futorny, Molev and Molev, Yakimova).
- We quantise not only the algebra $\overline{C_\xi}$ but also the argument shift ∂_ξ.

Derivation on $S\mathfrak{gl}(d,\mathbb{C})$

Let us consider $\mathfrak{g} = \mathfrak{gl}(d,\mathbb{C})$.

- Let
$$e = \begin{pmatrix} e_1^1 & \dots & e_d^1 \\ \dots & \dots & \dots \\ e_1^d & \dots & e_d^d \end{pmatrix} \in M\big(d, \mathfrak{gl}(d,\mathbb{C})\big),$$
where e_j^i form a linear basis of $\mathfrak{gl}(d,\mathbb{C})$ and satisfy the commutation relation $\left[e_j^i, e_l^k\right] = e_j^k \delta_l^i - \delta_j^k e_l^i$.
- A constant vector field on the dual space is given by
$$\partial_\xi = \operatorname{tr}(\xi\partial), \qquad \partial_j^i = \frac{\partial}{\partial e_i^j},$$
where ξ is a numerical matrix.

Derivation on $S\mathfrak{gl}(d, \mathbb{C})$

Remark

The derivation

$$S\mathfrak{gl}(d, \mathbb{C}) \to M\big(d, S\mathfrak{gl}(d, \mathbb{C})\big), \qquad x \mapsto \partial x = (\partial^i_j x)^d_{i,j=1}$$

is a unique linear mapping satisfying the following.

1. We have $\partial \nu = 0$ for any scalar ν.
2. We have $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. We have the Leibniz rule

$$\partial(xy) = (\partial x)y + x(\partial y)$$

for any elements x and y of the symmetric algebra.

Quantum Derivation on $U\mathfrak{gl}(d, \mathbb{C})$

- There is no such derivation ∂ on the universal enveloping algebra $U\mathfrak{gl}(d, \mathbb{C})$ since we obtain a contradiction

$$\begin{aligned} 0 &= \partial\left(e^i_j e^k_l - e^k_l e^i_j\right) && \text{(Leibniz rule)} \\ &= \partial\left(e^k_j \delta^i_l - \delta^k_j e^i_l\right) && \text{(commutation relation)} \\ &= E^k_j \delta^i_l - \delta^k_j E^i_l && \text{(second conditon)} \\ &\neq 0. \end{aligned}$$

- Gurevich, Pyatov, and Saponov defined the quantum derivation on the universal enveloping algebra.

Quantum Derivation on $U\mathfrak{gl}(d, \mathbb{C})$

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quantum derivation

$$U\mathfrak{gl}(d, \mathbb{C}) \to M\big(d, U\mathfrak{gl}(d, \mathbb{C})\big), \qquad x \mapsto \partial x = (\partial^i_j x)^d_{i,j=1}$$

is a unique linear mapping satisfying the following.

1. We have $\partial \nu = 0$ for any scalar ν.
2. We have $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. We have the quantum Leibniz rule

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y)$$

for any elements x and y of the universal enveloping algebra.

Quantum M-F Theorem

Theorem (I. and Sharygin, 2023)

Suppose that ξ *is a numerical matrix and let* $\partial_\xi = \operatorname{tr}(\xi\partial)$. *We have*

$$\left[\partial_\xi^m(x), \partial_\xi^n(y)\right] = 0$$

for any m *and* n *and for any central elements* x *and* y *of the universal enveloping algebra* $U\mathfrak{gl}(d, \mathbb{C})$.

arXiv: 2307.15952

Fundamental Formula

We assume the following form

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left(g_m^{(n-1)}(e)_j (e^m)^i + h_m^{(n-1)}(e)(e^m)^i_j \right),$$

where $g_m^{(n-1)}$ and $h_m^{(n-1)}$ are polynomials. We have

$$\partial(e^{n+1})^i_j = \sum_{m=0}^{n-1} \left(g_m^{(n-1)}(e)e + h_m^{(n-1)}(e) \right)_j (e^m)^i + \delta_j (e^n)^i$$
$$+ \sum_{m=0}^{n-1} \left(g_m^{(n-1)}(e)e^m \delta^i_j + h_m^{(n-1)}(e)(e^{m+1})^i_j \right)$$

by the quantum Leibniz rule and the commutation relation.

Fundamental Formula

We obtain the following recursion formulae:

1. $g_m^{(n)}(x) = g_m^{(n-1)}(x)x + h_m^{(n-1)}(x)$ for $0 \leq m < n$
2. $g_n^{(n)}(x) = 1$ for $0 \leq n$
3. $h_0^{(n)}(x) = \sum_{m=0}^{n-1} g_m^{(n-1)}(x)x^m$ for $0 \leq n$
4. $h_m^{(n)}(x) = h_{m-1}^{(n-1)}(x)$ for $0 < m \leq n$

And the solutions to them are

$$g_m^{(n)}(x) = f_+^{(n-m)}(x), \qquad h_m^{(n)}(x) = f_-^{(n-m)}(x),$$

where

$$f_\pm^{(n)}(x) = \frac{(x+1)^n \pm (x-1)^n}{2} = \sum_{m=0}^{n} \frac{1 \pm (-1)^{n-m}}{2} \binom{n}{m} x^m.$$

Fundamental Formula

We obtain the fundamental formula for the quantum derivation.

Theorem (I, 2022)

We have the formula

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left(f_+^{(n-m-1)}(e)_j (e^m)^i + f_-^{(n-m-1)}(e)(e^m)^i_j \right)$$

for any nonnegative integer n.

Fundamental Formula

The center C of the universal enveloping algebra $U\mathfrak{gl}(d, \mathbb{C})$

$$C \simeq \mathbb{C}\left[(\operatorname{tr} e^n)_{n=1}^d\right]$$

is a free commutative algebra on the set $(\operatorname{tr} e^n)_{n=1}^d$.

Definition

We define $C_\xi^{(n)} = C\left[\partial_\xi x, \ldots, \partial_\xi^n x : x \text{ is central}\right]$.

Remark

$C_\xi = \lim_{n\to\infty} C_\xi^{(n)}$ is the quantum argument shift algebra.

arXiv: 2309.15684

Corollary

We have $C_\xi^{(1)} = C\left[\operatorname{tr}(\xi e^n) : n \in \mathbb{N}\right]$.

Generators of Second Quantum Argument Shift

The main theorem of my talk is the following.

Theorem (I, 2023)

We have

$$C_\xi^{(2)} = C_\xi^{(1)}\left[\tau_\xi\left(P_n^{(n)}\right), \tau_\xi\left(P_{n+1}^{(n)} + P_n^{(n+1)}\right) : n \in \mathbb{N}\right].$$

Generators of Second Quantum Argument Shift

Using the fundamental formula twice we obtain

$$\partial_\xi^2(\operatorname{tr} e^n) + C_\xi^{(1)} = \sum_{m=-1}^{n} \operatorname{tr} e^m \sum_{k=-1}^{n-m-1} \operatorname{tr}\left(\xi\left(\partial \operatorname{tr}\left(\xi f_-^{(n-m-k-2)}(e)\right)\right) f_-^{(k)}(e)\right) + C_\xi^{(1)}. \quad (1)$$

We adopt the convention $\operatorname{tr} e^{-1} = f_-^{(-1)}(x) = 1$.

Key Matrix and Symmetry

Definition

1. We define P_n as the n by n submatrix of the following matrix from the bottom left corner.

$$\begin{pmatrix} \vdots \\ f_+^{(4)}(x) \\ f_+^{(3)}(x) \\ f_+^{(2)}(x) \\ f_+^{(1)}(x) \\ f_+^{(0)}(x) \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & 6 & 0 & 1 & \cdots \\ 0 & 3 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \\ \vdots \end{pmatrix}$$

2. We define $P_n^{(m)}$ as the matrix P_n shifted to the right by m positions.

Key Matrix and Symmetry

Definition

We define

$$\tau_\xi(A) = \mathrm{tr}\left(\begin{pmatrix} \xi & \xi e & \cdots & \xi e^{n-1} \end{pmatrix} A \begin{pmatrix} \xi \\ \xi e \\ \vdots \\ \xi e^{n-1} \end{pmatrix} \right)$$

for any n by n numerical matrix A.

We have

$$\mathrm{tr}\Big(\xi\big(\partial\,\mathrm{tr}(\xi e^n)\big)e^m\Big) + C_\xi^{(1)} = \tau_\xi(P_n^{(m)}) + C_\xi^{(1)} \tag{2}$$

by the fundamental formula.

Key Matrix and Symmetry

We obtain

$$C_{\xi}^{(2)} = C_{\xi}^{(1)} \left[\tau_{\xi} \left(P_{n}^{(m)} + P_{m}^{(n)} \right) : m, n \in \mathbb{N} \right]$$

by the equations (1) and (2).

Key Matrix and Symmetry

Definition

We define

$$\sigma(A) = \begin{pmatrix} A_1^1 & A_2^1 + A_1^2 & \cdots & A_n^1 + A_1^n \\ 0 & A_2^2 & \cdots & A_n^2 + A_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^n \end{pmatrix}$$

for any n by n numerical matrix A.

Lemma

We have $\tau_\xi(\sigma(A)) = \tau_\xi(A)$.

Main Theorem (Matrix Form)

Theorem (I, 2023)

We have

$$\sigma \begin{pmatrix} 0 & P_{m+2n} \\ P_m^T & 0 \end{pmatrix} = \sum_{k=0}^{n} \left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right) P_{m+k}^{(m+k)}$$

and

$$\sigma \begin{pmatrix} 0 & P_{m+2n+1} \\ P_m^T & 0 \end{pmatrix} = \sum_{k=0}^{n} \binom{2n-k}{k} \left(P_{m+k+1}^{(m+k)} + P_{m+k}^{(m+k+1)} \right)$$

for any nonnegative integers m and n.

Equivalent Formulae (Even Case)

The first formula of the theorem is equivalent to the following.

1. We have

$$\binom{2n_1+n_2+2n_3+1}{2n_3}+\binom{n_2+2n_3}{2n_3} = \sum_{n_4=0}^{n_3}\left(\binom{n_1+n_2+n_3+n_4+1}{2n_4}+\binom{n_1+n_2+n_3+n_4}{2n_4}\right)\binom{n_1+n_3-n_4}{2(n_3-n_4)}$$

for any nonnegative integers $(n_k)_{k=1}^{3}$.

Equivalent Formulae (Even Case)

2. We have

$$f_+^{(m+2n)}(x) + f_+^{(m)}(x)x^{2n} = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right)x^k f_+^{(m+k)}(x)$$

for any nonnegative integers m and n.

Equivalent Formulae (Even Case)

These formulae can be verified with Mathematica:

Main.nb

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In[1]:= FullSimplify[Binomial[2 n + m + 2 l + 1, 2 l] + Binomial[m + 2 l, 2 l] -
          Sum[(Binomial[n + m + l + k + 1, 2 k] + Binomial[n + m + l + k, 2 k])
            Binomial[n + l - k, 2 (l - k)], {k, 0, l}],
        Element[n | m | l, Integers] && n ≥ 0 && m ≥ 0 && l ≥ 0]

Out[1]= 0

In[2]:= Fplus[n_][x_] := ((x + 1) ^ n + (x - 1) ^ n) / 2

In[3]:= Simplify[Fplus[m + 2 n][x] + Fplus[m][x] x ^ (2 n) -
          Sum[(Binomial[2 n - k, k] + Binomial[2 n - k - 1, k - 1])
            Fplus[m + k][x] x ^ k, {k, 0, n}],
        Element[m | n, Integers] && m ≥ 0 && n ≥ 0]

Out[3]= 0
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Conclusion

- We replace the derivation on the symmetric algebra $S\mathfrak{gl}(d,\mathbb{C})$ by the quantum derivation on the universal enveloping algebra $U\mathfrak{gl}(d,\mathbb{C})$ in order to formulate the quantum analogue of the Mishchenko-Fomenko theorem.
- We can calculate iterated quantum argument shifts of central elements by the formula.
- The list of generators for the second order quantum argument shifts could be alternative to the lists suggested by Molev and others.

References

[1] Dimitri Gurevich, Pavel Pyatov, and Pavel Saponov. "Braided Weyl algebras and differential calculus on $U(u(2))$". In: *Journal of Geometry and Physics* 62.5 (2012), pp. 1175–1188.

[2] Yasushi Ikeda. "Quasidifferential operator and quantum argument shift method". In: *Theoretical and Mathematical Physics* 212.1 (2022), pp. 918–924.

[3] Yasushi Ikeda. "Second Quantum Argument Shifts in General Linear Lie Algebras". in preparation.

[4] Yasushi Ikeda, Alexander Molev, and Georgy Sharygin. *On the quantum argument shift method*. 2023. arXiv: 2309.15684 [math.RT].

[5] Yasushi Ikeda and Gerogy Sharygin. *The argument shift method in universal enveloping algebra $U\mathfrak{gl}_d$*. 2023. arXiv: 2307.15952 [math.RT].