$\mathfrak{g} = \mathfrak{gl}_d \subset U = U(\mathfrak{gl}_d)$

universal enveloping algebra

$Z(U)$: center of U

§ The matrix of generators

$$e = \begin{pmatrix} e^1_1 & \cdots & e^1_d \\ \vdots & & \\ e^d_1 & \cdots & e^d_d \end{pmatrix} \in M(d, U),$$

$$e^i_j = j\begin{pmatrix} & \overset{i}{} & \\ & 1 & \\ & & \end{pmatrix}$$

matrix unit

$\{\operatorname{tr} e, \ldots, \operatorname{tr} e^d\}$ generates $Z(U)$

§ Quasi-differentiation

$i, j = 1, \dots, d$ $\quad \exists_1 \, \partial^i_j : U \to U$ s.t. $\mathbb{R}$-linear

1) $\partial^i_j 1 = 0$

2) $\partial^i_j e^q_p = \delta^i_p \delta^q_j$

3) $\partial^i_j (fg) = (\partial^i_j f) g + f (\partial^i_j g) + \sum_{k=1}^{d} (\partial^i_k f)(\partial^k_j g)$

(modefied Leibniz rule)

Thm (Y.I.) $\quad \delta$: identity matrix

$$\partial^i_j \operatorname{tr} e^n = \Bigg(\underbrace{\frac{(e+\delta)^n - (e-\delta)^n}{2} + \sum_{k=0}^{n-1} (\operatorname{tr} e^k) \frac{(e+\delta)^{n-k-1} - (e-\delta)^{n-k-1}}{2}}_{= A(n)} \Bigg)^i_j$$

$$A(n) \in \operatorname{span}_{Z(U)} \{ e^k \}_{k=0}^{n}$$

$$\forall \xi \in \mathfrak{g}, \quad \partial_\xi \overset{\text{def}}{=} \sum_{i,j=1}^{d} \xi^j_i \, \partial^i_j$$

<u>Cor</u> 1) $\forall f \in Z(U)$,

$$\partial_\xi f \in \mathrm{span}_{Z(U)} \{\mathrm{tr}(\xi e^k)\}_{k=0}^{\infty}$$

2) $\forall f, \forall g \in Z(U)$, $\forall \xi \in \mathfrak{g}$,

$$[\partial_\xi f, \partial_\xi g] = 0$$

(Thm $\Rightarrow$ Cor) 1) WMA $f = \mathrm{tr}\, e^{n_1} \cdots \mathrm{tr}\, e^{n_m}$

$$\partial^i_j (\cancel{\mathrm{tr}\, e^{n_1} \cdots \mathrm{tr}\, e^{n_m}}) \quad f$$

$$= \sum_{p=1}^{m} \sum_{1 \le \alpha_1 < \cdots < \alpha_p \le m} \underbrace{\prod_{k \notin \{\alpha_1, \dots, \alpha_p\}} \mathrm{tr}\, e^{n_k}}_{\substack{= \\ Z_\alpha \\ \cap \\ Z(U)}}$$

the number of differentiation

$$\sum_{k_1, \dots, k_{p-1}=1}^{d} \underbrace{\boxed{\partial^i_{k_1} \mathrm{tr}\, e^{n_{\alpha_1}}}}_{} \partial^{k_1}_{k_2} \mathrm{tr}\, e^{n_{\alpha_2}} \cdots \partial^{k_{p-2}}_{k_{p-1}} \mathrm{tr}\, e^{n_{\alpha_{p-1}}} \partial^{k_{p-1}}_{j} \mathrm{tr}\, e^{n_{\alpha_p}}$$

$$\boxed{\partial^i_{k_1} \mathrm{tr}\, e^{n_{\alpha_1}}} = A(n_{\alpha_1})^i_{k_1}$$

$$= \sum_{p=1}^{m} \sum_{1 \le \alpha_1 < \cdots < \alpha_p \le m} \underset{\substack{\cap \\ Z(U)}}{Z_\alpha} \underset{\substack{\cap \\ \mathrm{span}_{Z(U)} \{e^k\}_{k=0}^{\infty}}}{(A(n_{\alpha_1}) \cdots A(n_{\alpha_p}))^i_j}$$

$$\Rightarrow \partial_\xi \left(\cancel{\mathrm{tr}\, e^{n_1} \cdots \mathrm{tr}\, e^{n_m}} \right)$$

$$\in \mathrm{span}_{Z(U)} \{ \mathrm{tr}(\xi e^k) \}_{k=0}^{\infty}$$

2) ISTS $[\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^n)] = 0$,

$\forall m, \forall n = 0, 1, 2, \ldots$

$$[\mathrm{tr}(\xi e^m), e_{ij}] = [e^m, \xi]_{ij}$$

$$\Rightarrow [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^n)]$$

$$= \sum_{k=1}^{n} \mathrm{tr}(\xi e^{k-1} [e^m, \xi] e^{n-k})$$

$$= \sum_{k=1}^{n} \mathrm{tr}([\xi e^{m+n-k}, \xi e^{k-1}])$$

WMA $m < n$

$$[\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^n)] = \frac{1}{2} \sum_{k=m+1}^{n} \mathrm{tr}([\xi e^{m+n-k}, \xi e^{k-1}])$$

$$= \frac{1}{2} \mathrm{tr}([\xi e^{n-1}, \cancel{\xi} e^m] + \cdots + [\xi e^m, \cancel{\xi} e^{n-1}])$$

$$= 0$$

§ Proof of Theorem

$$\partial^i_j (e^n)^p_q = (e^{n-1})^i_q \delta^p_j + \sum_{r=1}^{d} e^p_r \, \partial^i_j (e^{n-1})^r_q + \partial^p_j (e^{n-1})^i_q$$

$n > 0$

Lemma $\exists a^{(n)}_{k,l}, \exists b^{(n)}_{k,l} \in \mathbb{Z}_{\geq 0}$, s.t.

$$\partial^i_j (e^n)^p_q = \sum_{\substack{k,l \geq 0 \\ k+l < n}} \left(a^{(n)}_{k,l} (e^k)^i_q (e^l)^p_j + b^{(n)}_{k,l} (e^k)^p_q (e^l)^i_j \right)$$

$$= (e^{n-1})^i_q \delta^p_j + \sum_{k,l} a^{(n-1)}_{k,l} \left((e^k)^i_q (e^{l+1})^p_j + \delta^p_q (e^{k+l})^i_j \right)$$

$$+ \sum_{k,l} b^{(n-1)}_{k,l} \left((e^{k+1})^p_q (e^l)^i_j + (e^k)^i_q (e^l)^p_j \right)$$

Proof. Induction on n $\square$

$$\cdots \to \left(a_{k,l}^{(n-1)}, b_{k,l}^{(n-1)}\right)_{k,l}$$

$$\to \left(a_{k,l}^{(n)}, b_{k,l}^{(n)}\right)_{k,l} \to$$

Lemma $\Rightarrow \partial_j^i \operatorname{tr} e^n$

$$= \overbrace{\sum_{k,l} a_{k,l}^{(n)} \left(e^{k+l}\right)}^{= f^{(n)}(e)}{}_j^i$$

$$+ \sum_{k<n} \operatorname{tr} e^k \underbrace{\sum_{l} b_{k,l}^{(n)} \left(e^{l}\right)}_{= g_k^{(n)}(e)}{}_j^i$$

$$= f^{(n)}(e)_j^i + \sum_{k<n} \left(\operatorname{tr} e^k\right) g_k^{(n)}(e)_j^i ,$$

where $f^{(n)}(x) = \sum_{k,l} a_{k,l}^{(n)} x^{k+l}$ $\in \mathbb{Z}[x]$

$$g_k^{(n)}(x) = \sum_{l} b_{k,l}^{(n)} x^{l}$$

Lemma

$$\Rightarrow \begin{cases} f^{(n)}(x) = x^{n-1} + x f^{(n-1)}(x) + \sum_{k<n-1} x^k g_k^{(n-1)}(x) \\ g_0^{(n)}(x) = f^{(n-1)}(x) \\ g_{k+1}^{(n)}(x) = g_k^{(n-1)}(x), \quad 0 \le k < n-1 \end{cases}$$

$$= \cdots = g_0^{(n-k-1)}(x) = f^{(n-k-2)}(x)$$

$$f^{(0)}(x) = 0, \quad f^{(1)}(x) = 1,$$

$$f^{(n)}(x) = x^{n-1} + x f^{(n-1)}(x) + \sum_{k=0}^{n-2} x^k f^{(n-k-2)}(x),$$

$$n > 1$$

Solution

$$f^{(n)}(x) = \frac{(x+1)^n - (x-1)^n}{2}$$

$$\partial_j^i \operatorname{tr} e^n = \left(f^{(n)}(e) + \sum_{k=0}^{n-1} (\operatorname{tr} e^k) f^{(n-k-1)}(e) \right)_j^i$$

$$= \left(\frac{(e+\delta)^n - (e-\delta)^n}{2} + \sum_{k=0}^{n-1} (\operatorname{tr} e^k) \frac{(e+\delta)^{n-k-1} - (e-\delta)^{n-k-1}}{2} \right)_j$$

$\blacksquare$