Functional Analysis

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Contents

Chapter 1

Basic Theory

1.1 Locally Convex Spaces

Remark 1.1.1.

1. Suppose that $(f_i \colon X \to Y_i)_i$ is a family of continuous mappings. There exists a unique continuous mapping $\prod_i f_i \colon X \to \prod_i Y_i$ such that $f_i = p_i \circ \prod_i f_i$ for $\forall i$.
2. $\prod_i G_i$ is a topological group for a family $\forall (G_i)_i$ of topological groups.
3. Suppose that $(f_i \colon G \to H_i)_i$ is a family of continuous homomorphisms of topological groups. $\prod_i f_i$ is the unique continuous homomorphism such that $f_i = p_i \circ \prod_i f_i$ for $\forall i$.

Theorem 1.1.1. *Suppose that X is a finite dimensional complex (resp. real) vector space. There exists a unique locally convex topology on X.*

Remark 1.1.2.

1. A subspace of a topological vector (resp. locally convex) space is a topological vector (resp. locally convex) space.
2. The closure of a subspace of a topological vector space is a subspace.
3. The quotient space of a topological vector space by a closed subspace is a topological vector space.
4. The quotient topological vector space of a locally convex space is a locally convex space. The quotient topology is generated by the sufficient set
$$\left\{ [x] \mapsto \|[x]\| = \inf_{[x']=[x]} \|x'\| : \|\cdot\| \text{ is a continuous seminorm} \right\}$$
of seminorms.
5. $\prod_i X_i$ is a topological vector space for a family $\forall (X_i)_i$ of topological vector spaces.

6. $\prod_i X_i$ is a locally convex space and the topology of $\prod_i X_i$ is generated by the sufficient set
$$\bigcup_i \{\, x \mapsto \|x_i\|_i : \|\cdot\|_i \in S_i \,\}$$
of seminorms on $\prod_i X_i$ for a family $\forall(S_i)_i$ such that S_i is a sufficient set of seminorms on X_i generating the topology of X_i for $\forall i$ for a family $\forall(X_i)_i$ of locally convex spaces.

Remark 1.1.3.
1. A convex set of a locally convex space is closed if and only if it is weakly closed.
2. The weak topology of a quotient locally convex space is the quotient topology of the weak topology.
3. The weak topology of a product locally convex space is the product topology of the weak topologies.

Remark 1.1.4.
1. The dual space of a closed subspace of a locally convex (resp. normed vector) space is the quotient space of the dual space by the polar set and the weak* topology is the quotient topology of the weak* topology.
2. The dual space of a quotient locally convex (resp. normed vector) space is the polar set and the weak* topology is the relative weak* topology.
3. The dual space of a product locally convex space is the algebraic direct sum of the dual spaces and the weak* topology is the relative product topology of the weak* topologies.

Remark 1.1.5. A linear mapping of topological vector spaces is continuous if and only if it is continuous at 0.

Remark 1.1.6. The closed convex hull of a countable set of a locally convex space is separable.

Remark 1.1.7. If F is a subset of a locally convex space such that $K = \overline{\operatorname{conv}}\, F$ is compact then $\operatorname{ext} K$ is a subset of $\overline{F}$.

Remark 1.1.8.
$$\overline{\operatorname{conv}}(K_1 \cup \cdots \cup K_n) = \operatorname{conv}(K_1 \cup \cdots \cup K_n)$$
for compact convex sets $\forall K_1, \ldots, \forall K_n$ of a locally convex space.

Remark 1.1.9. The sum of a closed set and a compact set of a topological vector space is closed.

Remark 1.1.10. The subspace generated by a closed subspace and a finite subset of a locally convex space is closed.

Remark 1.1.11. A topological vector space is separable if and only if it is the closed subspace generated by a countable subset.

1.2 Normed Vector Spaces

Theorem 1.2.1. *Suppose that X is a finite dimensional complex (resp. real) vector space. There exists a unique norm on X up to equivalence.*

Remark 1.2.1. Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on a finite dimensional complex (resp. real) vector space.

$$\{ x : \|x\|_1 \leq 1 \} \simeq \{ x : \|x\|_2 \leq 1 \}.$$

Remark 1.2.2. A Cauchy sequence of a metric space converges if there exists a convergent subsequence.

Remark 1.2.3. A normed vector space is a Banach space if and only if $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists for a sequence $\forall (a_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|a_n\| < \infty$.

Remark 1.2.4. $|(x, y)| = \|x\|\|y\|$ if and only if x and y are linearly dependent for $\forall x$ and $\forall y$ of an inner product space.

Remark 1.2.5.

1. A subspace of a normed vector space is a normed vector space.
2. The quotient locally convex space of a normed vector (resp. Banach) space is a normed vector (resp. Banach) space with respect to the norm

$$\|[x]\| = \inf_{[x']=[x]} \|x'\|.$$

Remark 1.2.6. Suppose that X and Y are normed vector spaces.

1. $(x, y) \mapsto \|(x, y)\| = \|x\| + \|y\|$ is a norm on $X \times Y$.
2. $X \times Y$ is complete if and only if X and Y are complete.

Remark 1.2.7. Suppose that A is a linear mapping of normed vector spaces. The following are equivalent.

1. A is continuous at some point.
2. A is bounded.
3. A is uniformly continuous.

Proposition 1.2.1. *$B(X, Y)$ is a Banach (resp. normed vector) space for a normed vector space $\forall X$ and for a Banach (resp. normed vector) space $\forall Y$.*

Theorem 1.2.2 (Uniform Boundedness Theorem)**.** *A pointwise bounded subset of $B(X, Y)$ is bounded for a Banach space $\forall X$ and for a normed vector space $\forall Y$.*

Theorem 1.2.3 (Banach-Steinhaus Theorem)**.** *$x \mapsto \lim_{n\to\infty} A_n(x)$ is a bounded linear mapping for a sequence $\forall (A_n)_{n=1}^{\infty}$ of bounded linear mappings of a Banach space into a normed vector space such that $\lim_{n\to\infty} A_n(x)$ exists for $\forall x$.*

Theorem 1.2.4 (Open Mapping Theorem)**.** *A surjective bounded linear mapping of Banach spaces is an open mapping.*

Definition 1.2.1. Suppose that A is a bounded linear mapping of a normed vector space X into a normed vector space Y. The bounded linear mapping

$$A^*\colon Y^* \to X^*, \qquad f \mapsto f \circ A$$

is called the adjoint of A.

Remark 1.2.8. $\|A^*\| = \|A\|$.

Proposition 1.2.2. *A normed vector space is a weakly* dense subspace of the second dual space.*

Theorem 1.2.5. *A normed vector space is reflexive if and only if the unit ball is weakly compact.*

Theorem 1.2.6. *The unit ball of the dual space of a separable normed vector space is weakly* metrizable.*

Corollary 1.2.1. *The dual space of a separable normed vector space is weakly* separable.*

Theorem 1.2.7. *A weakly* compact set of the dual space of a Banach space is bounded.*

Corollary 1.2.2. *A weakly compact set of a normed vector space is bounded.*

1.3 Hahn-Banach Theorem

Definition 1.3.1. A real-valued function p on a real vector space X is called a sublinear functional if it satisfies the following.

1. $p(tx) = tp(x)$ for $\forall t \geq 0$ and for $\forall x$ of X.
2. $p(x+y) \leq p(x) + p(y)$ for $\forall x$ and $\forall y$ of X.

Theorem 1.3.1 (Hahn-Banach Theorem for Real Vector Spaces)**.** *Suppose that p is a sublinear functional on a real vector space X. A linear functional f_0 on a subspace X_0 such that $f_0 \leq p$ on X_0 extends to a linear functional f on X such that $f \leq p$ on X.*

Remark 1.3.1. A complex-valued real linear function f on a complex vector space is complex linear if $f(ix) = if(x)$ for $\forall x$.

Theorem 1.3.2 (Hahn-Banach Theorem for Vector Spaces)**.** *A linear functional on a subspace of a real or complex vector space dominated by a seminorm extends to a linear functional on the whole space dominated by the seminorm.*

Definition 1.3.2. A subset A of a real vector space X is said to be absorbing if $X = \bigcup_{t>0} tA$.

Remark 1.3.2. $x \mapsto \inf\{\, t > 0 : x \in tC \,\}$ is a sublinear functional for a convex absorbing set $\forall C$.

Theorem 1.3.3 (Hahn-Banach Theorem for Topological Vector Spaces)**.** *Suppose that O is an open convex set and C is a convex set of a topological vector space such that $O \cap C$ is empty. There exists a continuous linear functional f such that $\operatorname{Re} f(O)$ is a subset of $(-\infty, \inf \operatorname{Re} f(C))$.*

Theorem 1.3.4 (Hahn-Banach Theorem for Locally Convex Spaces)**.** *Suppose that F is a closed convex set and K is a compact convex set of a locally convex space such that $F \cap K$ is empty. There exists a continuous linear functional f such that $\sup \operatorname{Re} f(F) < \inf \operatorname{Re} f(K)$.*

Proposition 1.3.1. *A continuous linear functional on a subspace of a locally convex space extends to a continuous linear functional.*

Theorem 1.3.5 (Hahn-Banach Theorem for Normed Vector Spaces)**.** *A bounded linear functional f_0 on a subspace of a normed vector space extends to a bounded linear functional f on the whole space such that $\|f\| = \|f_0\|$.*

Theorem 1.3.6. *Suppose that Y is a subspace of a normed vector space and that x_0 is a vector such that $\inf\|x_0 - Y\| > 0$. There exists a bounded linear functional f such that Y is a subset of $\ker f$ and*

$$f(x_0) = \inf\|x_0 - Y\|, \qquad\qquad \|f\| = 1.$$

Corollary 1.3.1. *Suppose that x_0 is a nonzero vector of a normed vector space. There exists a bounded linear functional f such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.*

Chapter 2

Advanced Theory

2.1 Gelfand-Pettis Integral

Definition 2.1.1. A mapping f of a measurable space into a locally convex space A is said to be Gelfand-Pettis measurable if $\sigma \circ f$ is measurable for a continuous linear functional $\forall \sigma$ on A.

Definition 2.1.2. A Gelfand-Pettis measurable mapping f of a measure space μ into a locally convex space A is said to be Gelfand-Pettis integrable if there exists

$$\int f \, d\mu$$

of A such that $\sigma \circ f$ is integrable and

$$\sigma \int f \, d\mu = \int \sigma \circ f \, d\mu$$

for a continuous linear functional $\forall \sigma$ on A.

Remark 2.1.1. A weakly* Gelfand-Pettis measurable mapping f of a measure space (X, μ) into the dual space of a Banach space is weakly* Gelfand-Pettis integrable if $f(\cdot)(\sigma)$ is integrable for $\forall \sigma$ and the linear functional

$$\sigma \mapsto \int f(x)(\sigma)\mu(dx)$$

is bounded.

2.2 Bochner Integral

Suppose that (X, μ) is a complete measure space and that A is a Banach space.

Definition 2.2.1. A mapping f of X into A is said to be Bochner measurable if there exists a sequence $(f_n)_{n=1}^{\infty}$ of simple mappings of X into A such that $\lim_{n\to\infty} f_n = f$ a.e.

Definition 2.2.2. A mapping f of X into A is said to be Bochner integrable if there exists a sequence $(f_n)_{n=1}^{\infty}$ of simple mappings of X into A satisfying the following.

1. $\mu((f_n)^{-1}(A \setminus \{0\})) < \infty$ for $\forall n$.
2. $\|f_n - f\|$ is measurable for $\forall n$ and

$$\lim_{n\to\infty} \int \|f_n - f\| \, d\mu = 0.$$

Proposition 2.2.1. *A Bochner integrable mapping is Bochner measurable.*

Remark 2.2.1.

1. A Bochner measurable (resp. integrable) mapping is Gelfand-Pettis measurable (resp. integrable).
2. A function is Bochner measurable (resp. integrable) if and only if it is measurable (resp. integrable).

Proposition 2.2.2. *A mapping f of a σ-finite complete measure space into a Banach space is Bochner measurable if and only if there exists a sequence $(f_n)_{n=1}^{\infty}$ of Bochner integrable simple mappings such that* $\lim_{n\to\infty} f_n = f$ *a.e.*

Proposition 2.2.3. *A mapping f of a σ-finite complete measure space into a Banach space is Bochner measurable if there exists a sequence $(f_n)_{n=1}^{\infty}$ of Bochner measurable mappings such that* $\lim_{n\to\infty} f_n = f$ *a.e.*

Theorem 2.2.1 (Pettis Measurability Theorem)**.** *A Gelfand-Pettis measurable mapping f of a σ-finite complete measure space into a Banach space is Bochner measurable if and only if there exists a null set N such that $f(X \setminus N)$ is separable.*

Proposition 2.2.4. *A continuous mapping of the completion of a σ-finite Borel measure space on a separable locally compact Hausdorff space into a Banach space is Bochner measurable.*

Proposition 2.2.5.

$$\|\int f \, d\mu\| \leq \int \|f\| \, d\mu$$

for a Bochner integrable mapping $\forall f$.

Proposition 2.2.6. *A Bochner measurable mapping is Bochner integrable if and only if the norm is integrable.*

Definition 2.2.3. We denote the vector space of equivalence classes of elements of the vector space of Bochner measurable mappings by $M(\mu, A)$.

Remark 2.2.2. $M(\mu, A)$ is a complex algebra (resp. *-algebra) if A is a Banach algebra (resp. Banach *-algebra).

Proposition 2.2.7.

$$L^p(\mu, A) = \left\{ f \in M(\mu, A) : \|f\|_p = (\int \|f\|^p \, d\mu)^{1/p} < \infty \right\}$$

is a Banach space for $1 \leq \forall p < \infty$*.*

Remark 2.2.3. $L^p(\mu, \mathbb{C}) = L^p(\mu)$ for $1 \leq \forall p < \infty$.

Proposition 2.2.8. *Suppose that* H *is a Hilbert space.*

1. $L^2(\mu, H)$ *is a Hilbert space with respect to the inner product*

$$(f, g) = \int (f(x), g(x))\mu(dx).$$

2. $L^2(\mu, H) \simeq L^2(\mu) \otimes H$*.*

Proposition 2.2.9. *Suppose that* μ *is* σ*-finite.*

$$L^\infty(\mu, A) = \left\{ f \in M(\mu, A) : \|f\|_\infty = \min\{\, m : \|f\| \leq m \,\} \right.$$
$$\left. = \min_{\mu(N)=0} \sup_{x \in X \setminus N} \|f(x)\| < \infty \right\}$$

is a Banach space.

Remark 2.2.4.

1. $L^\infty(\mu, \mathbb{C}) = L^\infty(\mu)$.
2. $L^\infty(\mu, A)$ is a Banach algebra (resp. Banach *-algebra) if A is a Banach algebra (resp. Banach *-algebra).
3. $L^\infty(\mu, A)$ is a C^*-algebra if A is a C^*-algebra.

2.3 Vector Measures

Definition 2.3.1. A mapping of a σ-algebra into a locally convex space is called a vector measure if it is countably additive.

Proposition 2.3.1. *The composition of a continuous linear functional and a vector measure is a complex measure.*

2.4 Complex Analysis

Definition 2.4.1. A mapping f of an open set U of a complex (resp. real) normed vector space X into a complex (resp. real) topological vector space Y

is said to be differentiable at a point x_0 of U if there exists a continuous linear mapping $f^{(1)}(x_0)$ of X into Y such that

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0) - f^{(1)}(x_0)(x_n - x_0)}{\|x_n - x_0\|} = 0 \tag{2.1}$$

for a sequence $\forall(x_n)_{n=1}^{\infty}$ of $U \setminus \{x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$. In this case f is continuous at x_0 and $f^{(1)}(x_0)$ is unique.

Proposition 2.4.1. *A mapping f of an open set U of $\mathbb{C}$ (resp. $\mathbb{R}$) into a complex (resp. real) topological vector space is differentiable at a point x_0 of U if and only if*

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{2.2}$$

exists. In this case

$$f^{(1)}(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{2.3}$$

Proposition 2.4.2. *fg is differentiable at x_0 and*

$$(fg)^{(1)}(x_0) = f^{(1)}(x_0)g(x_0) + f(x_0)g^{(1)}(x_0)$$

for mappings $\forall f$ and $\forall g$ of an open set U of $\mathbb{C}$ (resp. $\mathbb{R}$) into a normed algebra differentiable at a point x_0 of U.

Proposition 2.4.3. *(f, g) is differentiable at x_0 and*

$$(f, g)^{(1)}(x_0) = (f^{(1)}(x_0), g(x_0)) + (f(x_0), g^{(1)}(x_0))$$

for mappings $\forall f$ and $\forall g$ of an open set U of $\mathbb{R}$ into a Hilbert space differentiable at a point x_0 of U.

Definition 2.4.2. A mapping of an open set U of $\mathbb{C}$ into a complex topological vector space is said to be holomorphic if it is differentiable at $\forall z$ of U.

Theorem 2.4.1. *Suppose that f is a mapping of an open set U of $\mathbb{C}$ into a complex Banach space A. The following are equivalent.*

1. *f is holomorphic.*
2. *Suppose that z_0 is a point of U. There exist $R > 0$ and a sequence $(a_n)_{n=1}^{\infty}$ of A such that*

   $$f(z) = \lim_{n\to\infty} \sum_{k=0}^{n} (z - z_0)^k a_k \tag{2.4}$$

   for $\forall z$ of $B(z_0, R)$.
3. *$\sigma \circ f$ is holomorphic for a bounded linear functional $\forall\sigma$ on A.*

In this case $(\sigma \circ f)^{(n)} = \sigma \circ f^{(n)}$ for a bounded linear functional $\forall\sigma$ on A and for $\forall n = 0, 1, \ldots$.

Theorem 2.4.2. *A mapping f of an open set U into the dual space of a complex Banach space such that $f(\cdot)(\sigma)$ is holomorphic for $\forall\sigma$ is holomorphic.*

Theorem 2.4.3. *Suppose that f is a holomorphic mapping of the open disk of radius $R > 0$ centered at a point z_0 of $\mathbb{C}$ into a complex Banach space.*

$$\sum_{n=0}^{\infty} \frac{|z - z_0|^n}{n!} \|f^{(n)}(z_0)\| < \infty \tag{2.5}$$

and

$$f(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) \tag{2.6}$$

for $\forall z$ of $B(z_0, R)$.

Proposition 2.4.4. *Suppose that f is a continuous mapping of the image of a piecewise continuously differentiable curve C of the complex plane into a complex Banach space A. There exists a unique*

$$\int_C f(z)\, dz \tag{2.7}$$

of A such that

$$\sigma \int_C f(z)\, dz = \int_C (\sigma \circ f)(z)\, dz \tag{2.8}$$

for a bounded linear functional $\forall\sigma$ on A.

Theorem 2.4.4 (Phragmén-Lindelöf)**.** *Suppose that*

$$D = \left\{ z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2} \right\} \tag{2.9}$$

and that f is a mapping into a Banach space continuous on $\overline{D}$ holomorphic on D.

$$\sup_{z \in D} \|f(z)\| = \sup_{z \in \partial D} \|f(z)\| \tag{2.10}$$

if there exists $0 \leq C < 1$ such that

$$\sup_{z \in D} \frac{\|f(z)\|}{\exp(\cosh(C \operatorname{Re} z))} < \infty. \tag{2.11}$$

Theorem 2.4.5. *Suppose that f is a mapping into a Banach space bounded continuous on $\mathbb{R} \times [0, 1]$ holomorphic on $\mathbb{R} \times (0, 1)$.*

$$\sup_{\operatorname{Im} z = y} \|f(z)\| \leq (\sup_{\operatorname{Im} z = 0} \|f(z)\|)^{1-y} (\sup_{\operatorname{Im} z = 1} \|f(z)\|)^{y} \leq \sup_{\operatorname{Im} z = 0,1} \|f(z)\|$$

for $0 \leq \forall y \leq 1$.

Corollary 2.4.1. *Suppose that f is a mapping into a Banach space bounded continuous on $\mathbb{R} \times [0,1]$ holomorphic on $\mathbb{R} \times (0,1)$.*

$$\sup_{0<\operatorname{Im} z<1} \|f(z)\| = \sup_{\operatorname{Im} z=0,1} \|f(z)\|.$$

Corollary 2.4.2. *Suppose that f is a mapping into a Banach space bounded continuous on $\mathbb{R} \times [0,1]$ holomorphic on $\mathbb{R} \times (0,1)$. If*

$$\sup_{\operatorname{Im} z=0} \|f(z)\| = 0$$

then $f = 0$.

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