Differential Equations

Yasushi Ikeda

November 27, 2022

Suppose that f is a continuous mapping of an open set U of $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$. And we suppose that each point (t_0, x_0) of U has a neighborhood V contained in U such that there exists a function φ of class C^1 on the open set

$$W = \{\, (t, x, y) \in \mathbb{R}^{2n+1} : (t, x), (t, y) \in V \,\}$$

satisfying the following.

1. The equation
$$\varphi(t, x, x) = 0$$
holds for each point (t, x) of V and the relation
$$\varphi(t, x, y) > 0$$
holds provided that $x \neq y$ for each point (t, x, y) of W.

2. The relation
$$\frac{\partial\varphi(t, x, y)}{\partial t} + \frac{\partial\varphi(t, x, y)}{\partial x} f(t, x) + \frac{\partial\varphi(t, x, y)}{\partial y} f(t, y) \leq 0$$
holds for each point (t, x, y) of W.

Suppose that (t_0, x_0) is a point of U. A partially ordered set consisting of x that is a differentiable mapping of an open subinterval of $[t_0, \infty)$ containing t_0 into $\mathbb{R}^n$ such that $x(t_0) = x_0$ and

$$\frac{dx(t)}{dt} = f\bigl(t, x(t)\bigr)$$

for each t is a totally ordered set and has a maximum x [1, Section 2.6]. Suppose that Δ is a compact subset of U. Then the point $\bigl(t, x(t)\bigr)$ belongs to $U \setminus \Delta$ eventually [1, Section 2.6].

Suppose that f is a continuous mapping of $[x_0, x_1]$ into $\mathbb{R}^n$. Then we have

$$\|\int_{x_0}^{x_1} f(x)\, dx\| \leq \int_{x_0}^{x_1} \|f(x)\|\, dx.$$

Suppose that f is a C^1 mapping of an open set U of $\mathbb{R}^n$ into $\mathbb{R}^n$. Suppose that x_0 and x_1 are points of U such that x_0x_1 is a subset of U. Then we have

$$\|f(x_1) - f(x_0)\| \leq (\max_{x \in x_0x_1} \|f_*(x)\|)\|x_1 - x_0\|.$$

Proof. The mapping $g(t) = f\bigl(tx_1 + (1-t)x_0\bigr)$ is of class C^1 on $[0,1]$. We have

$$\begin{aligned} f(x_1) - f(x_0) &= g(1) - g(0) \\ &= \int_0^1 \frac{dg(t)}{dt}\, dt \\ &= \int_0^1 f_*\bigl(tx_1 + (1-t)x_0\bigr)(x_1 - x_0)\, dt. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|f(x_1) - f(x_0)\| &\le \int_0^1 \|f_*\bigl(tx_1 + (1-t)x_0\bigr)(x_1 - x_0)\|\, dt \\ &\le (\max_{x \in x_0x_1} \|f_*(x)\|)\|x_1 - x_0\|. \end{aligned}$$

□

Suppose that X and Y are topological spaces. We denote the set of continuous mappings of X into Y by $C(X, Y)$.

Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We denote $C(X, \mathbb{F})$ by $C(X)$. Then $C(X)^d = C(X, \mathbb{F}^d)$ is a vector space over $\mathbb{F}$.

Suppose that X is a compact space. Then $C(X)^d$ is a Banach space over $\mathbb{F}$ with respect to the norm

$$\|f\| = \sup_{x \in X} \|f(x)\|.$$

The uniform space $C(X)^d$ is a closed subspace of the uniform space $(\mathbb{F}^d)^X$ with respect to the uniformity of uniform convergence. The sequence $\{f_n\}_{n=1}^\infty$ of $C(X)^d$ has a uniformly convergent subsequence if $\{f_n\}_{n=1}^\infty$ is equicontinuous and pointwise bounded (Arzelà-Ascoli).

Suppose that f is a bounded continuous mapping of

$$R = [t_0, t_0 + \delta] \times \bigl\{\, x \in \mathbb{R}^d : \|x - x_0\| \le \varepsilon \,\bigr\}$$

into $\mathbb{R}^d$ such that

$$(\sup_{(t,x) \in R} \|f(t,x)\|)\delta \le \varepsilon.$$

We define

$$\|f\| = \sup_{(t,x) \in R} \|f(t,x)\|.$$

Suppose that $t_0 < t_1 < \cdots < t_n = t_0 + \delta$. We define

$$x_k = x_{k-1} + f(t_{k-1}, x_{k-1})(t_k - t_{k-1})$$

for each k. Then the points (t_0, x_0), (t_1, x_1), …, (t_n, x_n) belong to R. We define x to be the unique function whose graph is $(t_0, x_0) \cdots (t_n, x_n)$. Since

$$\begin{aligned} &\max_k \sup_{(t,x) \in (t_{k-1}, x_{k-1})(t_k, x_k)} \|(t,x) - (t_{k-1}, x_{k-1})\| \\ &\qquad \le \max_k \sup_{(t,x) \in (t_{k-1}, x_{k-1})(t_k, x_k)} \sqrt{1 + \|f\|^2}(t - t_{k-1}) \\ &\qquad\qquad \le \sqrt{1 + \|f\|^2} \max_k (t_k - t_{k-1}) \end{aligned}$$

and f is uniformly continuous on R, we have

$$\lim_{\max_k(t_k-t_{k-1})\to 0} \max_k \sup_{(t,x)\in(t_{k-1},x_{k-1})(t_k,x_k)} \|f(t,x)-f(t_{k-1},x_{k-1})\| = 0. \qquad (1)$$

We define

$$\begin{aligned} r &= \max_k \sup_{(t,x)\in(t_{k-1},x_{k-1})(t_k,x_k)} \|f(t,x)-f(t_{k-1},x_{k-1})\| \\ &= \max_k \sup_{t\in[t_{k-1},t_k]} \Big\| f\big(t,x(t)\big) - \frac{dx(t)}{dt} \Big\|. \end{aligned}$$

Then we have

$$\Big\| \int_{t_0}^{t} f\big(t,x(t)\big)\,dt - x(t) + x_0 \Big\| \le r(t-t_0).$$

There exists a C^1 mapping x of $[t_0, t_0+\delta]$ into

$$\{\, x\in\mathbb{R}^d : \|x-x_0\| \le \varepsilon \,\}$$

such that $x(t_0)=x_0$ and

$$\frac{dx}{dt} = f(t,x)$$

on $[t_0, t_0+\delta]$.

Proof. We define

$$t_0 < \cdots < t_k = t_0 + \frac{k\delta}{n} < \cdots < t_n = t_0+\delta.$$

Then we have

$$\lim_{n\to\infty} \max_k (t_k - t_{k-1}) = 0.$$

We define

$$r_n = \max_k \sup_{(t,x)\in(t_{k-1},x_{k-1})(t_k,x_k)} \|f(t,x)-f(t_{k-1},x_{k-1})\|.$$

By the equation (1), we have $\lim_{n\to\infty} r_n = 0$. We denote the unique function whose graph is $(t_0,x_0)\cdots(t_n,x_n)$ by x_n. Then we have

$$\Big\| \int_{t_0}^{t} f\big(t,x_n(t)\big)\,dt - x_n(t) + x_0 \Big\| \le r_n(t-t_0).$$

The sequence $\{x_n\}_{n=1}^{\infty}$ is uniformly bounded and uniformly equicontinuous since

$$\|x_n(\tau_1) - x_n(\tau_2)\| \le \|f\| |\tau_1 - \tau_2|$$

for each τ_1 and τ_2. The sequence $\{x_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence by the Arzelà-Ascoli theorem. We denote the subsequence again by $\{x_n\}_{n=1}^{\infty}$. We define $x = \lim_{n\to\infty} x_n$. We have

$$\lim_{n\to\infty} \int_{t_0}^{t} f\big(t,x_n(t)\big)\,dt = \int_{t_0}^{t} f\big(t,x(t)\big)\,dt$$

by the bounded convergence theorem. We have

$$x(t) = x_0 + \int_{t_0}^{t} f\bigl(t, x(t)\bigr)\, dt$$

and

$$x(t_0) = x_0, \qquad\qquad \frac{dx(t)}{dt} = f\bigl(t, x(t)\bigr).$$

□

Suppose that A is a nonempty subset of a metric space X. Then we have

$$\bigl|\operatorname{dist}(x_1, A) - \operatorname{dist}(x_2, A)\bigr| \le \operatorname{dist}(x_1, x_2)$$

for each (x_1, x_2) and the mapping $x \mapsto \operatorname{dist}(x, A)$ is uniformly continuous.

Proof. Since we have

$$\operatorname{dist}(x_1, A) - \operatorname{dist}(x_2, a) \le \operatorname{dist}(x_1, a) - \operatorname{dist}(x_2, a) \le \operatorname{dist}(x_1, x_2)$$

for each point a of A, we have

$$\operatorname{dist}(x_1, A) - \operatorname{dist}(x_2, A) = \sup_{a \in A}\bigl(\operatorname{dist}(x_1, A) - \operatorname{dist}(x_2, a)\bigr) \le \operatorname{dist}(x_1, x_2).$$

Therefore, we have

$$\bigl|\operatorname{dist}(x_1, A) - \operatorname{dist}(x_2, A)\bigr| \le \operatorname{dist}(x_1, x_2)$$

and the mapping $x \mapsto \operatorname{dist}(x, A)$ is uniformly continuous. □

Suppose that U is an open set of a metric space X. Suppose that C is a compact subset of U. Then there exists $\delta > 0$ such that

$$C_\delta = \bigl\{\, x \in X : \operatorname{dist}(C, x) < \delta \,\bigr\}$$

is a subset of U.

Proof. We may assume that C and $X \setminus U$ are nonempty. Then we have

$$\delta = \operatorname{dist}(C, X \setminus U) = \min_{x \in C} \operatorname{dist}(x, X \setminus U) > 0$$

and C_δ is a subset of U. □

Suppose that f is a bounded continuous mapping of $[0,1] \times \mathbb{R}^d$ into $\mathbb{R}^d$. Suppose that x is a C^1 function on a subinterval of $[0,1]$ containing a neighborhood of 0 such that $x(0) = 0$ and

$$\frac{dx(t)}{dt} = f\bigl(t, x(t)\bigr)$$

on $\operatorname{dom} x$. Then the solution x can be extended to the whole interval $[0,1]$ [1, Section 5.2]. We define

$$S^{(0)} = \Big\{ x^{(0)} : x^{(0)} \text{ is a } C^1 \text{ function on } [0,1] \text{ such that} \\ x^{(0)}(0) = 0 \text{ and } \frac{dx^{(0)}(t)}{dt} = f\big(t, x^{(0)}(t)\big) \text{ on } [0,1] \Big\}.$$

and $C_{x^{(0)}}$ to be the graph of $x^{(0)}$ for each element $x^{(0)}$ of $S^{(0)}$. Then the set

$$C = \bigcup_{x^{(0)} \in S^{(0)}} C_{x^{(0)}}$$

is closed [1, Section 5.3].

Suppose that f is a bounded continuous mapping of $\mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}^d$. We define

$$S^{(t_0,x_0)} = \Big\{ x : x \text{ is a } C^1 \text{ function on } [t_0,1] \text{ such that} \\ x(t_0) = x_0 \text{ and } \frac{dx(t)}{dt} = f\big(t, x(t)\big) \text{ on } [t_0,1] \Big\}$$

for each point (t_0, x_0) of $(-\infty, 1) \times \mathbb{R}^d$. We assume that C is bounded (thus compact) and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that [there exists an element $x^{(0)}$ of $S^{(0)}$ such that

$$\max\Big\{ \operatorname{dist}\Big(C_{x^{(0)}}, \big(t, x(t)\big)\Big) : t_0 \le t \le 1 \Big\} < \varepsilon]$$

for each point (t_0, x_0) of $(-\infty, 1) \times \mathbb{R}^d$ and each element x of $S^{(t_0,x_0)}$ such that $\operatorname{dist}\big(C, (t_0, x_0)\big) < \delta$.

Proof. Suppose contrary. We may assume that $t_n < 1$ and x_n is an element of $S^{(t_n, x_n(t_n))}$ for each n and the following.

1. We have
$$1 > \operatorname{dist}\Big(C, \big(t_n, x_n(t_n)\big)\Big) \to 0.$$

2. Suppose that $x^{(0)}$ is an element of $S^{(0)}$. Then we have
$$\inf_n \max\Big\{ \operatorname{dist}\Big(C_{x^{(0)}}, \big(t, x_n(t)\big)\Big) : t_n \le t \le 1 \Big\} \ge \varepsilon.$$

We may assume that x_n belongs to $S^{(-1, x_n(-1))}$ for each n. Then we have

$$x_n(t) = x_n(t_n) + \int_{t_n}^{t} f\big(t, x_n(t)\big)\, dt$$

on $[-1,1]$ for each n and the sequence $(x_n)_{n=1}^\infty$ is uniformly bounded on $[-1,1]$. The sequence $(x_n)_{n=1}^\infty$ is uniformly equicontinuous on $[-1,1]$ since

$$\bigl\|x_n(\tau_2) - x_n(\tau_1)\bigr\| = \|\int_{\tau_1}^{\tau_2} f\bigl(t, x_n(t)\bigr)\,dt\| \le \|f\||\tau_2 - \tau_1|.$$

We may assume that $\lim_{n\to\infty} x_n = x$ uniformly by the Arzelà-Ascoli theorem and $\lim_{n\to\infty} t_n = t_0$. Then we have

$$\begin{aligned} x(t) &= \lim_{n\to\infty} x_n(t) \\ &= \lim_{n\to\infty} \left(x_n(0) + \int_0^t f\bigl(t, x_n(t)\bigr)\,dt \right) \\ &= x(0) + \int_0^t f\bigl(t, x(t)\bigr)\,dt \end{aligned}$$

on $[-1,1]$ and x belongs to $S^{(-1,x(-1))}$. The point $\bigl(t_0, x(t_0)\bigr)$ belongs to C since

$$x(t_0) = \lim_{n\to\infty} \bigl(x_n(t_n) - x(t_n) + x(t_n)\bigr) = \lim_{n\to\infty} x_n(t_n)$$

and

$$\operatorname{dist}\Bigl(C, \bigl(t_0, x(t_0)\bigr)\Bigr) = \lim_{n\to\infty} \operatorname{dist}\Bigl(C, \bigl(t_n, x_n(t_n)\bigr)\Bigr) = 0.$$

There exists an element $x^{(0)}$ of $S^{(0)}$ such that $x^{(0)} = x$ on $[t_0, 1]$. Then we have

$$\inf_n \max\Bigl\{ \operatorname{dist}\Bigl(C_{x^{(0)}}, \bigl(t, x_n(t)\bigr)\Bigr) : t_n \le t \le 1 \Bigr\} \ge \varepsilon.$$

This is a contradiction. □

Suppose that f is a continuous mapping of a neighborhood U of the origin $(0,0)$ of $\mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}^d$ and we assume the following.

1. Any C^1 function $x^{(0)}$ on a subinterval $\operatorname{dom} x^{(0)}$ of $[0,1]$ containing a neighborhood of 0 such that $x^{(0)}(0) = 0$ and

   $$\frac{dx^{(0)}(t)}{dt} = f\bigl(t, x^{(0)}(t)\bigr)$$

   on $\operatorname{dom} x^{(0)}$ extends to an element of $S^{(0)}$.

2. The closure of the set

   $$C = \bigcup_{x^{(0)} \in S^{(0)}} C_{x^{(0)}}$$

   is a compact subset of U.

Then the set C is closed and [there exists $\delta > 0$ satisfying the following] for any $\varepsilon > 0$.

1. The set
$$C_\delta = \Big\{ (t,x) : \operatorname{dist}\big(C,(t,x)\big) < \delta \Big\}$$
is contained in U.

2. Suppose that (t_0, x_0) is a point of $C_\delta \cap \big((-\infty,1)\times\mathbb{R}^d\big)$. Then any C^1 function x on a subinterval $\operatorname{dom} x$ of $[t_0,1]$ containing a neighborhood of t_0 such that $x(t_0)=x_0$ and
$$\frac{dx(t)}{dt} = f\big(t, x(t)\big)$$
on $\operatorname{dom} x$ extends to an element of $S^{(t_0,x_0)}$ and there exists an element $x^{(0)}$ of $S^{(0)}$ such that
$$\max\Big\{ \operatorname{dist}\Big(C_{x^{(0)}}, \big(t,x(t)\big)\Big) : t_0 \le t \le 1 \Big\} < \varepsilon.$$

Proof. There exists a compact subset Δ of U such that the closure of C is contained in the interior of Δ. Then there exists a bounded continuous mapping $\tilde{f}$ of $\mathbb{R}\times\mathbb{R}^d$ into $\mathbb{R}^d$ such that $\tilde{f} = f$ on Δ by the Tietze extension theorem. We define $\tilde{S}^{(t_0,x_0)}$ and $\tilde{C}$ related to $\tilde{f}$ in the same manner. Then the set $S^{(0)}$ is contained in $\tilde{S}^{(0)}$ and $\tilde{C}$ is a closed set containing C. Suppose that x is an element of $\tilde{S}^{(0)}$ and we assume that the compact set
$$(\partial\Delta)^{-1} = \big\{ t : \big(t, x(t)\big) \in \partial\Delta \big\}$$
is nonempty. We define $t_0 = \min(\partial\Delta)^{-1} > 0$. Then the connected space $[0, t_0)$ is a subset of
$$(\operatorname{Int}\Delta)^{-1} = \big\{ t : \big(t, x(t)\big) \in \operatorname{Int}\Delta \big\}.$$
The restriction of the function x to $[0,t_0]$ extends to an element of $S^{(0)}$. This is a contradiction and the function x belongs to $S^{(0)}$. Then we have $S^{(0)} = \tilde{S}^{(0)}$ and the set $C = \tilde{C}$ is closed (thus compact).

We may assume that
$$C_\varepsilon = \Big\{ (t,x) : \operatorname{dist}\big(C,(t,x)\big) < \varepsilon \Big\}$$
is contained in the interior of Δ. There exists $\delta > 0$ such that [there exists an element $x^{(0)}$ of $S^{(0)}$ such that
$$\max\Big\{ \operatorname{dist}\Big(C_{x^{(0)}}, \big(t,x(t)\big)\Big) : t_0 \le t \le 1 \Big\} < \varepsilon]$$
for each point (t_0,x_0) of $(-\infty,1)\times\mathbb{R}^d$ and each element x of $\tilde{S}^{(t_0,x_0)}$ such that $\operatorname{dist}\big(C,(t_0,x_0)\big) < \delta$. We may assume that
$$C_\delta = \Big\{ (t,x) : \operatorname{dist}\big(C,(t,x)\big) < \delta \Big\}$$

is contained in the interior of Δ. Suppose that (t_0, x_0) is a point of $C_\delta \cap \big((-\infty, 1) \times \mathbb{R}^d\big)$. We take any C^1 function x on a subinterval $\operatorname{dom} x$ of $[t_0, 1]$ containing a neighborhood of t_0 such that $x(t_0) = x_0$ and

$$\frac{dx(t)}{dt} = f\big(t, x(t)\big)$$

on $\operatorname{dom} x$. Suppose that

$$(\partial\Delta)^{-1} = \big\{\, t : \big(t, x(t)\big) \in \partial\Delta \,\big\}$$

is nonempty. We define $t_1 = \min(\partial\Delta)^{-1} > t_0$. Then the connected space $[t_0, t_1)$ is a subset of

$$(\operatorname{Int}\Delta)^{-1} = \big\{\, t : \big(t, x(t)\big) \in \operatorname{Int}\Delta \,\big\}.$$

Then there exists an element $x^{(0)}$ of $S^{(0)}$ such that

$$\max\Big\{\, \operatorname{dist}\Big(C_{x^{(0)}}, \big(t, x(t)\big)\Big) : t_0 \le t \le t_1 \,\Big\} < \varepsilon.$$

The point $\big(t_1, x(t_1)\big)$ belongs to the interior of Δ. This is a contradiction and the function x extends to an element of $\tilde{S}^{(t_0,x_0)}$. There exists an element $x^{(0)}$ of $S^{(0)}$ such that

$$\max\Big\{\, \operatorname{dist}\Big(C_{x^{(0)}}, \big(t, x(t)\big)\Big) : t_0 \le t \le 1 \,\Big\} < \varepsilon.$$

This means that the function x belongs to $S^{(t_0,x_0)}$. □

Suppose that f is a continuous mapping of a neighborhood U of the origin $(0, 0)$ of $\mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}^d$ and we assume the following.

1. Any C^1 function $x^{(0)}$ on a subinterval $\operatorname{dom} x^{(0)}$ of $[0, 1]$ containing a neighborhood of 0 such that $x^{(0)}(0) = 0$ and

   $$\frac{dx^{(0)}(t)}{dt} = f\big(t, x^{(0)}(t)\big)$$

   on $\operatorname{dom} x^{(0)}$ extends to an element of $S^{(0)}$.

2. The closure of the set

   $$C = \bigcup_{x^{(0)} \in S^{(0)}} C_{x^{(0)}}$$

   is a compact subset of U.

Then the set C is closed. Suppose that U_0 is a neighborhood of C. Then there exists a neighborhood V_0 of C satisfying [the following for each point (t_0, x_0) of $V_0 \cap (-\infty, 1) \times \mathbb{R}^d$].

1. Any solution for the Cauchy problem

   $$x(t_0) = x_0, \qquad \frac{dx(t)}{dt} = f\big(t, x(t)\big), \qquad \big(t, x(t)\big) \in U$$

   can be extended to $[t_0, 1]$.

2. The graph of any such extension to $[t_0, 1]$ is contained in U_0.

Suppose that f is a continuous mapping of an open set U of $\mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}^d$ and we assume that [there exists $\delta > 0$ such that there exists a unique C^1 function x such that $x(t_0) = x_0$ and

$$\frac{dx(t)}{dt} = f\big(t, x(t)\big)$$

on $(t_0 - \delta, t_0 + \delta)$] for each point (t_0, x_0) of U.

Suppose that $x^{(1)}$ and $x^{(2)}$ are C^1 functions on open intervals containing t_0 such that $x^{(i)}(t_0) = x_0$ and

$$\frac{dx^{(i)}}{dt} = f\big(t, x^{(i)}\big).$$

Then we have $x^{(1)} = x^{(2)}$ on $\operatorname{dom} x^{(1)} \cap \operatorname{dom} x^{(2)}$. A partially ordered set consisting of x that is a C^1 mapping on an open interval containing t_0 such that $x(t_0) = x_0$ and

$$\frac{dx}{dt} = f(t, x)$$

has a maximum $x[t_0, x_0]$.

Suppose that $x^{(0)}$ is a C^1 function on $[t_1, t_2]$ such that

$$\frac{dx^{(0)}}{dt} = f\big(t, x^{(0)}\big).$$

Then there exists a neighborhood V of $C^{(0)}$ such that [$\operatorname{dom} x[t_0, x_0]$ contains $[t_1, t_2]$ for each point (t_0, x_0) of V].

The function $x(t, t_0, x_0) = x[t_0, x_0](t)$ is continuous on the open set $\operatorname{dom} x$.

Proof. We define $x^{(0)} = x[0, 0]$. Suppose that $t_1 \geq 0$ is a point of $\operatorname{dom} x^{(0)}$. Then there exists $\delta_0 > 0$ such that $[-\delta_0, t_1 + \delta_0]$ is contained in $\operatorname{dom} x^{(0)}$. We define

$$M = \max\Big\{ \|\frac{dx^{(0)}(t)}{dt}\| : -\delta_0 \leq t \leq t_1 + \delta_0 \Big\} < \infty.$$

We denote the graph of $x^{(0)}$ on $[-\delta_0, t_1 + \delta_0]$ by $C^{(0)}$. Suppose that $\varepsilon > 0$. Then there exists $\delta > 0$ such that $x = x[t_0, x_0]$ is defined on $[-\delta_0, t_1 + \delta_0]$ and

$$\max\Big\{ \operatorname{dist}\Big(C^{(0)}, \big(t, x(t, t_0, x_0)\big)\Big) : -\delta_0 \leq t \leq t_1 + \delta_0 \Big\} < \varepsilon$$

for each point (t_0, x_0) of $C^{(0)}_\delta$. We may assume that $\delta \leq \delta_0$. Suppose that $\|(t_0, x_0)\| < \delta$. Then we have

$$\max\Big\{ \operatorname{dist}\Big(C^{(0)}, \big(t, x(t, t_0, x_0)\big)\Big) : -\delta_0 \leq t \leq t_1 + \delta_0 \Big\} < \varepsilon.$$

Suppose that $|t - t_1| < \delta$. Then we have

$$\operatorname{dist}\Big(C^{(0)}, \big(t, x(t, t_0, x_0)\big)\Big) < \varepsilon.$$

There exists a point t_2 of $[-\delta_0, t_1 + \delta_0]$ such that

$$\big\|\big(t_2, x^{(0)}(t_2)\big) - \big(t, x(t, t_0, x_0)\big)\big\| < \varepsilon.$$

Then we have

$$\begin{aligned}\|x(t, t_0, x_0) - x(t_1, 0, 0)\| &\le \|x(t, t_0, x_0) - x^{(0)}(t_2)\| + \|x^{(0)}(t_2) - x^{(0)}(t_1)\| \\ &< \varepsilon + M(\varepsilon + \delta) \\ &\le (2M + 1)\varepsilon.\end{aligned}$$

□

Suppose that $x^{(0)}$ is a C^1 function on $[0, t_1]$ and we define $C^{(0)}$ to be the graph of $x^{(0)}$. Suppose that V is a neighborhood of $C^{(0)}$ and φ is a C^1 function on the open set

$$W = \big\{\, (t, x, y) : (t, x), (t, y) \in V \,\big\}$$

such that the equation

$$\varphi(t, x, x) = 0$$

holds for each point (t, x) of V and the relation

$$\varphi(t, x, y) > 0$$

holds provided that $x \ne y$ for each point (t, x, y) of W. Then there exists $\delta > 0$ satisfying the following. Suppose that f is a continuous function on V such that

$$\frac{\partial \varphi(t, x, y)}{\partial t} + \frac{\partial \varphi(t, x, y)}{\partial x} f(t, x) + \frac{\partial \varphi(t, x, y)}{\partial y} f(t, y) \le 0$$

holds for each point (t, x, y) of W and

$$\int_0^{t_1} \Big\| \frac{dx^{(0)}(t)}{dt} - f\big(t, x^{(0)}(t)\big) \Big\|\, dt < \delta.$$

Suppose that (t_0, x_0) is a point of $C_\delta^{(0)} \cap [0, t_1) \times \mathbb{R}^d$. Then there exists a C^1 function x on $[t_0, t_1]$ such that $x(t_0) = x_0$ and

$$\frac{dx(t)}{dt} = f\big(t, x(t)\big)$$

on $[t_0, t_1]$.

Proof. There exists a compact subset Δ of V such that the set $C^{(0)}$ is contained in the interior of Δ. There exists $\delta > 0$ such that the set

$$C_\delta^{(0)} = \Big\{\, (t, x) : \operatorname{dist}\big(C, (t, x)\big) < \delta \,\Big\}$$

is contained in the interior of Δ. We define

$$t_2 = \min\big\{\, t : \big(t, x(t)\big) \in \partial\Delta \,\big\} > t_0.$$

We define

$$M = \max_{(t,x)\in\Delta\cap[0,t_1]\times\mathbb{R}^d} \Big\| \frac{\partial\varphi\big(t, x^{(0)}(t), x\big)}{\partial x} \Big\| < \infty.$$

Then we have

$$\begin{aligned}\frac{d\varphi\big(t, x^{(0)}(t), x(t)\big)}{dt} &= \frac{\partial\varphi\big(t, x^{(0)}(t), x(t)\big)}{\partial t} \\ &\quad + \frac{\partial\varphi\big(t, x^{(0)}(t), x(t)\big)}{\partial x}\frac{dx^{(0)}(t)}{dt} + \frac{\partial\varphi\big(t, x^{(0)}(t), x(t)\big)}{\partial y}\frac{dx(t)}{dt} \\ &\le \frac{\partial\varphi\big(t, x^{(0)}(t), x(t)\big)}{\partial x}\left(\frac{dx^{(0)}(t)}{dt} - f\big(t, x^{(0)}(t)\big)\right) \\ &\le M\Big\|\frac{dx^{(0)}(t)}{dt} - f\big(t, x^{(0)}(t)\big)\Big\|\end{aligned}$$

for $t_0 \le t \le t_3 = \min\{t_1, t_2\}$ and

$$\begin{aligned}\varphi\big(t_3, x^{(0)}(t_3), x(t_3)\big) - \varphi\big(t_0, x^{(0)}(t_0), x_0\big) &\le M\int_{t_0}^{t_3} \Big\|\frac{dx^{(0)}(t)}{dt} - f\big(t, x^{(0)}(t)\big)\Big\|\, dt \\ &\le M\delta.\end{aligned}$$

We define

$$\eta = \min_{(t,x)\in\partial\Delta\cap[0,t_1]\times\mathbb{R}^d} \varphi\big(t, x^{(0)}(t), x\big) > 0$$

and we may assume that

$$\varphi\big(t_0, x^{(0)}(t_0), x_0\big) < \eta - M\delta$$

for each point (t_0, x_0) of $C_\delta^{(0)} \cap [0, t_1) \times \mathbb{R}^d$.

$$\begin{aligned}\eta &\le \varphi\big(t_2, x^{(0)}(t_2), x(t_2)\big) \\ &\le \varphi\big(t_0, x^{(0)}(t_0), x_0\big) + M\delta \\ &< \eta\end{aligned}$$

provided that $t_2 \le t_1$. This is a contradiction and we have $t_2 > t_1$. $\square$

Suppose that f is a continuous mapping of an open set V of $\mathbb{R}\times\mathbb{R}^d$ into $\mathbb{R}^d$. Suppose that x is a C^1 function on $[0, 1]$ such that $x(0) = 0$ and

$$\frac{dx(t)}{dt} = f\big(t, x(t)\big)$$

on $[0,1]$. We denote the graph of x by C. Suppose that φ is a C^1 function on the open set

$$W = \{\, (t,x,y) : (t,x), (t,y) \in V \,\}$$

such that the equation

$$\varphi(t,x,x) = 0$$

holds for each point (t,x) of V and the relation

$$\varphi(t,x,y) > 0$$

holds provided that $x \neq y$ for each point (t,x,y) of W. Suppose that f_i is a net of continuous mappings of V into $\mathbb{R}^d$ such that

$$\frac{\partial\varphi(t,x,y)}{\partial t} + \frac{\partial\varphi(t,x,y)}{\partial x} f_i(t,x) + \frac{\partial\varphi(t,x,y)}{\partial y} f_i(t,y) \leq 0$$

holds for each point (t,x,y) of W for each i and we assume that

$$\lim_i \int_0^1 \big\| f\big(t,x(t)\big) - f_i\big(t,x(t)\big) \big\| \, dt = 0.$$

Then the solution for the Cauchy problem

$$x_i(0) = 0, \qquad \frac{dx_i(t)}{dt} = f_i\big(t, x_i(t)\big)$$

is defined on $[0,1]$ eventually and x_i converges to x uniformly on $[0,1]$.

Proof. Suppose that $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\int_0^1 \big\| f\big(t,x(t)\big) - f_i\big(t,x(t)\big) \big\| \, dt < \delta$$

implies that the solution for the Cauchy problem

$$x_i(0) = 0, \qquad \frac{dx_i(t)}{dt} = f_i\big(t, x_i(t)\big)$$

is defined on $[0,1]$ and $\|x_i - x\| < \varepsilon$. □

Suppose that

$$x(0) = 0, \qquad \frac{dx(t)}{dt} = f(t)x(t) + g(t)$$

is a Cauchy problem of a linear differential equation. Suppose that f_i and g_i are nets of continuous functions on $[0,1]$ such that $\lim_i f_i = f$ and $\lim_i g_i = g$ uniformly on $[0,1]$. Suppose that x_i is a solution for the Cauchy problem

$$x_i(0) = 0, \qquad \frac{dx_i(t)}{dt} = f_i(t)x_i(t) + g_i(t).$$

Then we have $\lim_i x_i = x$ uniformly on $[0,1]$.

Proof. Since

$$\|f_i(t)x + g_i(t) - f_i(t)y - g_i(t)\| \le \|f_i\|\|x - y\|,$$

the differential equations satisfy the Lipschitz condition. Since

$$\int_0^1 \|f(t)x(t) + g(t) - f_i(t)x(t) - g_i(t)\|\, dt \le \|f - f_i\|\|x\| + \|g - g_i\| \to 0,$$

$\lim_i x_i = x$ uniformly. ☐

Suppose that f is a C^1 mapping of an open set U of $\mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}^d$. We define $x = x[t_0, x_0]$. Suppose that e is a unit vector and we define $x_\delta(t) = x(t, \delta e) = x(t, t_0, x_0 + \delta e)$. Then the function x_δ is defined on each compact subset of $\operatorname{dom} x$ eventually and $\lim_{\delta\to 0} x_\delta = x$ compactly on $\operatorname{dom} x$. We define $\delta x = x_\delta - x$. Then we have

$$\begin{aligned}\frac{d\delta x(t)}{dt} &= f\big(t, x_\delta(t)\big) - f\big(t, x(t)\big)\\ &= \left(\int_0^1 \frac{\partial f\big(t, x(t) + \theta\delta x(t)\big)}{\partial x}\, d\theta\right)\delta x(t)\end{aligned}$$

and we have

$$\frac{\delta x(t_0)}{\delta} = e, \qquad \frac{d}{dt}\frac{\delta x(t)}{\delta} = \left(\int_0^1 \frac{\partial f\big(t, x(t) + \theta\delta x(t)\big)}{\partial x}\, d\theta\right)\frac{\delta x(t)}{\delta}.$$

Since

$$\lim_{\delta\to 0}\int_0^1 \frac{\partial f\big(t, x(t) + \theta\delta x(t)\big)}{\partial x}\, d\theta = \frac{\partial f\big(t, x(t)\big)}{\partial x}$$

compactly on $\operatorname{dom} x$, we have

$$\lim_{\delta\to 0}\frac{\delta x(t)}{\delta} = \frac{\partial x(t, t_0, x_0)}{\partial e}$$

compactly and we have

$$\frac{\partial x(t_0, t_0, x_0)}{\partial e} = e, \qquad \frac{\partial}{\partial t}\frac{\partial x(t, t_0, x_0)}{\partial e} = \frac{\partial f\big(t, x(t, t_0, x_0)\big)}{\partial x}\frac{\partial x(t, t_0, x_0)}{\partial e}.$$

The function

$$(t, t_0, x_0) \mapsto \frac{\partial x(t, t_0, x_0)}{\partial e}$$

is continuous.

Proof. We have

$$\lim_{(t_0, x_0)\to 0} x[t_0, x_0] = x[0, 0]$$

compactly on $\operatorname{dom} x[0,0]$. We have

$$\lim_{(t_0,x_0)\to 0} \frac{\partial f\big(t, x(t,t_0,x_0)\big)}{\partial x} = \frac{\partial f\big(t, x(t,0,0)\big)}{\partial x}$$

compactly on $\operatorname{dom} x[0,0]$. We have

$$\lim_{(t_0,x_0)\to 0} \frac{\partial x(t,t_0,x_0)}{\partial e} = \frac{\partial x(t,0,0)}{\partial e}$$

compactly on $\operatorname{dom} x[0,0]$. □

The function $(t,t_0,x_0) \mapsto x(t,t_0,x_0)$ is of class C^1 and

$$\frac{\partial x(t_0,t_0,x_0)}{\partial x_0} = 1, \qquad \frac{\partial}{\partial t}\frac{\partial x(t,t_0,x_0)}{\partial x_0} = \frac{\partial f\big(t, x(t,t_0,x_0)\big)}{\partial x}\frac{\partial x(t,t_0,x_0)}{\partial x_0},$$

$$\frac{\partial x(t,t_0,x_0)}{\partial t_0} = -\frac{\partial x(t,t_0,x_0)}{\partial x_0} f(t_0,x_0), \qquad \frac{\partial x(t,t_0,x_0)}{\partial t} = f\big(t, x(t,t_0,x_0)\big).$$

Suppose that f is a smooth mapping of an open U of $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d'}$ into $\mathbb{R}^d$. We define $x(t) = x(t,t_0,x_0,c)$ by

$$x(t_0) = x_0, \qquad \frac{dx(t)}{dt} = f\big(t, x(t), c\big).$$

Then the function $(t,t_0,x_0,c) \mapsto x(t,t_0,x_0,c)$ is smooth and

$$\frac{\partial x(t_0,t_0,x_0,c)}{\partial x_0} = 1, \quad \frac{\partial}{\partial t}\frac{\partial x(t,t_0,x_0,c)}{\partial x_0} = \frac{\partial f\big(t, x(t,t_0,x_0,c),c\big)}{\partial x}\frac{\partial x(t,t_0,x_0,c)}{\partial x_0},$$

$$\frac{\partial x(t,t_0,x_0,c)}{\partial t_0} = -\frac{\partial x(t,t_0,x_0,c)}{\partial x_0} f(t_0,x_0,c),$$

$$\frac{\partial x(t,t_0,x_0,c)}{\partial t} = f\big(t, x(t,t_0,x_0,c),c\big),$$

$$\frac{\partial x(t_0,t_0,x_0,c)}{\partial c} = 0,$$

$$\frac{\partial}{\partial t}\frac{\partial x(t,t_0,x_0,c)}{\partial c} = \frac{\partial f\big(t, x(t,t_0,x_0,c),c\big)}{\partial x}\frac{\partial x(t,t_0,x_0,c)}{\partial c} + \frac{\partial f\big(t, x(t,t_0,x_0,c),c\big)}{\partial c}.$$

References

[1] Hiroshi Okamura. *Introduction to differential equations (in Japanese)*. Kawade Shobo, 1950.