Quantum Argument Shifts in General Linear Lie Algebras

Yasushi Ikeda

Moscow State University

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Outline

Prior Research and Motivation

- Let $(e_n)_{n=1}^{d}$ be a linear basis of a complex Lie algebra $\mathfrak{g}$.
- The symmetric algebra $S\mathfrak{g}$ has a unique Poisson bracket extending the Lie bracket

$$\begin{tikzcd} S\mathfrak{g} \times S\mathfrak{g} \arrow[r, "\text{Poisson bracket}"] & S\mathfrak{g} \\ \mathfrak{g} \times \mathfrak{g} \arrow[u] \arrow[r, "\text{Lie bracket}"] & \mathfrak{g} \arrow[u] \end{tikzcd}.$$

Prior Research and Motivation

Let $\overline{C}$ be the Poisson center of the Poisson algebra $S\mathfrak{g}$. Suppose that ξ is an element of the dual space $\mathfrak{g}^*$ and let $\overline{\partial}_\xi = \sum_{n=1}^{d} \xi(e_n) \frac{\partial}{\partial e_n}$. The following is referred to as the argument shift method.

Theorem (A. Mishchenko and A. Fomenko, 1978)

The subset $\left\{ \overline{\partial}_\xi^n x : (n, x) \in \mathbb{N} \times \overline{C} \right\}$ *is Poisson commutative.*

Prior Research and Motivation

- We obtained the $\boxed{\text{Poisson commutative}}$ subalgebra $\overline{C}_\xi$ generated by the elements $\overline{\partial}_\xi^n x$.
- The universal enveloping algebra $U\mathfrak{g}$ is considered as the quantisation of the symmetric algebra $S\mathfrak{g}$ and we have $\boxed{\operatorname{gr} U\mathfrak{g} = S\mathfrak{g}}$.
- Vinberg asked if the Poisson commutative subalgebra $\overline{C}_\xi$ can be quantised to the $\boxed{\text{commutative}}$ subalgebra C_ξ of the universal enveloping algebra $U\mathfrak{g}$ with $\boxed{\operatorname{gr} C_\xi = \overline{C}_\xi}$.

Prior Research and Motivation

- Vinberg's problem is solved by
  - Nazarov and Olshanski: (twisted) Yangians.
  - Tarasov: symmetrisation mapping.
- Vinberg's problem is also solved by the Feigin-Frenkel center
  - for regular elements ξ (Feigin et al. and Rybnikov).
  - for simple Lie algebras of types A and C (Futorny, Molev and Molev, Yakimova).

Motivation

The purpose of my talk is to quantise not only the algebra $\overline{C}_\xi$ but also the operator $\overline{\partial}_\xi$.

Quantum Derivation of Algebra $U\mathfrak{gl}_d$

Let $e = \begin{pmatrix} e^1_1 & \cdots & e^1_d \\ \vdots & \ddots & \vdots \\ e^d_1 & \cdots & e^d_d \end{pmatrix}$ be a linear basis of the general linear Lie algebra $\mathfrak{gl}_d$ satisfying the commutation relations

$$\left[e^i_j, e^k_\ell\right] = \delta^i_\ell e^k_j - e^i_\ell \delta^k_j.$$

We define

$$\bar{\partial}^i_j = \frac{\partial}{\partial e^j_i}, \qquad \bar{\partial} x = \begin{pmatrix} \bar{\partial}^1_1 x & \cdots & \bar{\partial}^1_d x \\ \vdots & \ddots & \vdots \\ \bar{\partial}^d_1 x & \cdots & \bar{\partial}^d_d x \end{pmatrix}$$

for any element x of the symmetric algebra $S\mathfrak{gl}_d$.

Quantum Derivation of Algebra $U\mathfrak{gl}_d$

Remark

The derivation

$$S\mathfrak{gl}_d \to M(d, S\mathfrak{gl}_d), \qquad x \mapsto \overline{\partial} x$$

is a unique linear mapping satisfying the following.

1. $\overline{\partial}\nu = 0$ for any scalar ν.
2. $\overline{\partial}\operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (Leibniz rule)

$$\overline{\partial}(xy) = (\overline{\partial}x)y + x(\overline{\partial}y)$$

   for any elements x and y of the symmetric algebra $S\mathfrak{gl}_d$.

Quantum Derivation of Algebra $U\mathfrak{gl}_d$

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quantum derivation

$$\boxed{U\mathfrak{gl}_d} \to M(d, U\mathfrak{gl}_d), \qquad x \mapsto \partial x$$

is a unique linear mapping satisfying the following.

1. $\partial \nu = 0$ for any scalar ν.
2. $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. (quantum Leibniz rule)

   $$\partial(xy) = (\partial x)y + x(\partial y) + \boxed{(\partial x)(\partial y)}$$

   for any elements x and y of the universal enveloping algebra $U\mathfrak{gl}_d$.

Quantum Derivation of Algebra $U\mathfrak{gl}_d$

Let C be the center of the algebra $U\mathfrak{gl}_d$. Suppose that ξ is a numerical matrix and let $\partial_\xi = \operatorname{tr}(\xi\partial)$. The main theorem is the following.

Theorem (I. and Sharygin, 2023)

The subset

$$\left\{ \partial_\xi^n x : (n, x) \in \mathbb{N} \times C \right\} \tag{1}$$

is commutative.

Corollary

The subalgebra C_ξ generated by the subset (1) is the quantum argument shift algebra in the direction ξ.

Quantum Derivation of Algebra $U\mathfrak{gl}_d$

- We may assume that $\xi = \operatorname{diag}(z_1, \ldots, z_d)$ is diagonal and $(z_1, \ldots, z_d)$ is distinct considering the adjoint action of the general linear Lie group GL_d.
- Vinberg and Rybnikov showed that the quantum argument shift algebra in the direction ξ is the centraliser of the set

$$\left\{ e_i^i, \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j} \right\}_{i=1}^{d}.$$

- The proof is carried out by showing that these elements commute with the quantum argument shift $\partial_\xi^n x$ by induction on the natural number n.

Formula for Quantum Derivation

We have

$$C = \mathbb{C}\left[\operatorname{tr} e, \dots, \operatorname{tr} e^d\right]$$

and the center C of the universal enveloping algebra $U\mathfrak{gl}_d$ is the free commutative algebra on the elements

$$\operatorname{tr} e, \qquad \dots, \qquad \operatorname{tr} e^d.$$

They are called the Gelfand invariants. We would like to calculate the quantum argument shift $\partial_\xi^n x$ for a central element x. It is necessary and even sufficient to calculate the quantum derivation $\partial(e^n)^i_j$.

Formula for Quantum Derivation

I obtained the following formula for the quantum derivation.

Definition

We define $f_{\pm}^{(m)}(x) = \dfrac{(x+1)^m \pm (x-1)^m}{2}$.

Theorem (I, 2022)

We have the formula

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left(f_{+}^{(n-m-1)}(e)_j (e^m)^i + f_{-}^{(n-m-1)}(e)(e^m)^i_j \right)$$

for any nonnegative integer n.

The formula is used for the base case.

Formula for Quantum Derivation

Proof.

We assume the following form

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left(g_m^{(n-1)}(e)_j (e^m)^i + h_m^{(n-1)}(e)(e^m)^i_j \right),$$

where $g_m^{(n-1)}$ and $h_m^{(n-1)}$ are polynomials. We have

$$\partial(e^{n+1})^i_j = \sum_{k=1}^{d} \partial\left((e^n)^i_k e^k_j \right) = \cdots$$

by the quantum Leibniz rule and the commutation relations.

Formula for Quantum Derivation

Proof.

We obtain the recursion formulae

1. $g_m^{(n)}(x) = g_m^{(n-1)}(x)x + h_m^{(n-1)}(x)$ for $0 \leq m < n$.
2. $g_n^{(n)}(x) = 1$ for $0 \leq n$.
3. $h_0^{(n)}(x) = \sum_{m=0}^{n-1} g_m^{(n-1)}(x)x^m$ for $0 \leq n$.
4. $h_m^{(n)}(x) = h_{m-1}^{(n-1)}(x)$ for $0 < m \leq n$.

The solution to them turn out to be

$$g_m^{(n)}(x) = f_+^{(n-m)}(x), \qquad h_m^{(n)}(x) = f_-^{(n-m)}(x).$$

$\square$

Formula for Quantum Derivation

We write ∂ for the algebraic homomorphism

$$U\mathfrak{gl}_d \to M(d, U\mathfrak{gl}_d), \qquad x \mapsto \operatorname{diag}(x, \dots, x) + \partial x$$

from now on. We write $C_\xi^{(n)}$ for the subalgebra generated by the subset $\bigcup_{m=0}^{n} \partial_\xi^m C$. We have

$$\begin{aligned} &\partial_\xi \left(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \cdots \right) \\ &\qquad = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots \operatorname{tr}\Big(\xi \prod_k f_-^{(n_k - m_k - 1)}(e)\Big) \end{aligned}$$

and the subalgebra $C_\xi^{(1)}$ is generated by the subset

$$C \cup \left\{ \operatorname{tr}\left(\xi e^n\right) \right\}_{n=0}^{\infty}.$$

Proof of Main Theorem

We show the base case $n = 1$. We are reduced to show

$$\left[\sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \operatorname{tr}\left(\xi e^n\right)\right] = \sum_{j\neq i} \frac{1}{z_i - z_j} \sum_{k=1}^{d} z_k \left[e_i^j e_j^i, (e^n)_k^k\right]$$

$$= \sum_{j=1}^{d} \left(-(e^n)_i^j e_j^i + e_i^j (e^n)_j^i\right) = 0.$$

To make the inductive step work it is sufficient to show

$$\left[\operatorname{ad} e_i^i, \partial_\xi\right] = \left[\left[\operatorname{ad} \sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right], \partial_\xi\right] = 0.$$

It can be done by computation.

Generators of Quantum Argument Shift Algebra

$\bigcup_{n=0}^{\infty} C_{\xi}^{(n)}$ is the quantum argument shift algebra in the direction ξ. We have

$$\partial_{\xi}^{2}\big(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \cdots\big) = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots$$

$$\sum_{k_1=-1}^{n_1-m_1-1} \sum_{k_2=-1}^{n_2-m_2-1} \cdots \operatorname{tr}\Big(\xi \prod_{\ell} f_{-}^{(k_\ell)}(e) \partial \operatorname{tr}\big(\xi \prod_{\ell} f_{-}^{(n_\ell-m_\ell-k_\ell-2)}(e)\big)\Big)$$

and the subalgebra $C_{\xi}^{(2)}$ is generated by the subset

$$C \cup \Big\{\operatorname{tr}\Big(\xi e^{m} \partial \operatorname{tr}\big(\xi e^{n}\big)\Big) + \operatorname{tr}\Big(\xi e^{n} \partial \operatorname{tr}\big(\xi e^{m}\big)\Big)\Big\}_{m,n=0}^{\infty}.$$

Generators of Quantum Argument Shift Algebra

Definition

We define the $m+n$ by n integer matrix $P_n^{(m)}$ by

$$x^m f_+^{(n-j)}(x) = \sum_{i=1}^{m+n} \left(P_n^{(m)}\right)_j^i x^{i-1}$$

and let $P_n = P_n^{(0)}$.

Definition

We define

$$\tau_\xi(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j^i \operatorname{tr}\left(\xi e^{i-1} \xi e^{j-1}\right)$$

for any m by n integer matrix x.

Generators of Quantum Argument Shift Algebra

Definition

We define the n by n lower triangular integer matrix $\sigma(x)$ by

$$\sigma(x) = \begin{pmatrix} x_1^1 & 0 & \cdots & 0 \\ x_1^2 + x_2^1 & x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n + x_n^1 & x_2^n + x_n^2 & \cdots & x_n^n \end{pmatrix}$$

for any n by n integer matrix x.

Generators of Quantum Argument Shift Algebra

Proposition

We have $\tau_\xi(\sigma(x)) = \tau_\xi(x)$ for any square integer matrix x.

Proof.

It is equivalent to the conditions $\mathrm{tr}(\xi e^m \xi e^n) = \mathrm{tr}(\xi e^n \xi e^m)$. $\square$

We have

$$\mathrm{tr}\left(\xi e^m \partial\, \mathrm{tr}(\xi e^n)\right) = \tau_\xi\left(P_n^{(m)}\right),$$

$$\mathrm{tr}\left(\xi e^m \partial\, \mathrm{tr}(\xi e^n)\right) + \mathrm{tr}\left(\xi e^n \partial\, \mathrm{tr}(\xi e^m)\right) = \tau_\xi\left(\sigma\begin{pmatrix} 0 & P_n^T \\ P_m & 0 \end{pmatrix}\right)$$

modulo $C_\xi^{(1)}$.

Generators of Quantum Argument Shift Algebra

The following indicates the linear dependence of the generators.

Theorem (I, 2023)

We have

$$\sigma\begin{pmatrix} 0 & P_m^T \\ P_{m+2n} & 0 \end{pmatrix} = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right) P_{m+k}^{(m+k)},$$

$$\sigma\begin{pmatrix} 0 & P_m^T \\ P_{m+2n+1} & 0 \end{pmatrix} = \sum_{k=0}^{n}\binom{2n-k}{k}\left(P_{m+k+1}^{(m+k)} + P_{m+k}^{(m+k+1)}\right).$$

for any nonnegative integers m and n.

Generators of Quantum Argument Shift Algebra

We obtained the conclusion.

Theorem (I, 2023)

The subalgebra $C_\xi^{(2)}$ is generated by the subalgebra $C_\xi^{(1)}$ and the subset

$$\left\{ \tau_\xi\left(P_n^{(n)}\right), \tau_\xi\left(P_{n+1}^{(n)}\right) + \tau_\xi\left(P_n^{(n+1)}\right) : n = 1, 2, \ldots \right\}.$$

Generators of Quantum Argument Shift Algebra

The generators are $\operatorname{tr}(\xi e)$, $\operatorname{tr}(\xi e^2)$, ... and

$$
\begin{aligned}
&\operatorname{tr}\left(\xi^2 e\right),\\
&\operatorname{tr}\left(2\xi^2 e^2 + \xi e \xi e\right),\\
&\operatorname{tr}\left(\xi^2 e^3 + \xi e \xi e^2\right),\\
&\operatorname{tr}\left(2\xi^2 e^4 + 2\xi e \xi e^3 + \xi e^2 \xi e^2 + \xi^2 e^2\right),\\
&\operatorname{tr}\left(\xi^2 e^5 + \xi e \xi e^4 + \xi e^2 \xi e^3 + \xi^2 e^3\right),\\
&\operatorname{tr}\left(2\xi^2 e^6 + 2\xi e \xi e^5 + 2\xi e^2 \xi e^4 + \xi e^3 \xi e^3 + 4\xi^2 e^4 + \xi e \xi e^3\right),\\
&\operatorname{tr}\left(\xi^2 e^7 + \xi e \xi e^6 + \xi e^2 \xi e^5 + \xi e^3 \xi e^4 + 3\xi^2 e^5 + \xi e \xi e^4\right), \ldots.
\end{aligned}
$$

They are mutually commutative.