§1 Introduction

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$\mathfrak{g}_d = \mathfrak{gl}(d, \mathbb{C})$

$\mathfrak{g}_d \subset \underbrace{S(\mathfrak{g}_d)}_{\text{sym. alg.}} \simeq \mathbb{C}[e_{ij} : i, j = 1, \dots, d]$

$\mathfrak{g}_d = \langle e_{ij} : i, j = 1, \dots, d \rangle$

$$\begin{array}{ccc} \mathfrak{g}_d & \simeq & M(d, \mathbb{C}) \\ \downarrow & & \downarrow \\ e_{ij} & \leftrightarrows & E_{ij} = j\begin{pmatrix} \overset{i}{} \\ & 1 \end{pmatrix} \end{array}$$

(j, i)-matrix unit

$$\begin{array}{ccc} S\mathfrak{g}_d \times S\mathfrak{g}_d & \xrightarrow[\{\cdot,\cdot\}\ \text{curly bracket}]{\exists_1\ \text{Poisson}} & S\mathfrak{g}_d \\ \cup & & \cup \\ \mathfrak{g}_d \times \mathfrak{g}_d & \xrightarrow[{[\cdot,\cdot]}\ \text{Lie}]{} & \mathfrak{g}_d \end{array}$$

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$\{\cdot,\cdot\} : S\mathfrak{g}_d \times S\mathfrak{g}_d \to S\mathfrak{g}_d$

is called the Lie-Poisson bracket.

$$\partial^i_j \underset{\text{def}}{=} \frac{\partial}{\partial e^j_i} \Rightarrow \partial^i_j e^k_\ell = \delta^i_\ell \delta^k_j$$

$\in \hom S\mathfrak{g}_d$

$\{f^j_i\}_{i,j}$: dual basis of $\{e^i_j\}^d_{i,j=1}$

$\cap$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\cap$

$\mathfrak{g}^*_d$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\mathfrak{g}_d$.

i.e $f^i_j(e^k_\ell) = \delta^i_\ell \delta^k_j$.

$$\forall \xi = \sum_{i,j=1}^d \xi^i_j f^j_i, \qquad \operatorname{tr}(\xi\partial)$$

$\cap$

$\mathfrak{g}^*_d$

$$\partial_\xi \underset{\text{def}}{=} \sum_{i,j=1}^d \xi^i_j \partial^j_i$$

$$e^i_j \in \mathfrak{g}_d \subset U\mathfrak{g}_d$$
$$\subset S\mathfrak{g}_d$$

$$e \underset{\text{def}}{=} \begin{pmatrix} e^1_1 & \cdots & e^1_d \\ \vdots & \ddots & \vdots \\ e^d_1 & \cdots & e^d_d \end{pmatrix} \cong \sum_{i,j=1}^{d} e^i_j \otimes E^j_i$$

$$\cap \qquad\qquad \cap$$

$$M(d, U\mathfrak{g}_d) \cong U\mathfrak{g}_d \otimes M(d, \mathbb{C})$$

$$(\text{or} \quad M(d, S\mathfrak{g}_d) \cong S\mathfrak{g}_d \otimes M(d, \mathbb{C}))$$

$C \underset{\text{def}}{=}$ center of $U\mathfrak{g}_d$

$\bar{C} \underset{\text{def}}{=}$ Poisson center of $S\mathfrak{g}_d$ (wrt Lie-Poisson bracket $\{\ ,\ \}$)

<u>Fact</u> $\operatorname{tr} e, \operatorname{tr} e^2, \cdots \in C$ (resp. $\bar{C}$)

and $\operatorname{tr} e, \cdots, \operatorname{tr} e^d$ freely generate C (resp. $\bar{C}$), i.e.

$$\begin{array}{ccccc} Ug_d & & d\text{-poly alg} & & Sg_d \\ \cup & & & & \cup \\ C & \simeq & \mathbb{C}[x_1, \dots, x_d] & \simeq & \overline{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathrm{tr}\, e^i & \simeq & x_i & \simeq & \mathrm{tr}\, e^i \end{array}$$

<u>Thm 1</u> (Mishchenko-Fomenko, 1978)

$\forall \xi \in g_d^*$, $\forall x, \forall y \in \overline{C}$, $\forall m, \forall n = 0, 1, 2, \dots$

$$\{\partial_\xi^m x, \partial_\xi^n y\} = 0.$$

<u>Def 1</u> (argument shift alg)

$$\overline{C_\xi} \underset{\text{def}}{=} \overline{C}[\partial_\xi x, \partial_\xi^2 x, \dots : x \in \overline{C}]$$

(subalg generated by

$$\overline{C} \cup \partial_\xi \overline{C} \cup \partial_\xi^2 \overline{C} \cup \cdots)$$

H-F Thm $\Rightarrow$ $\boxed{\overline{C_\xi} : \text{Poisson commutative}}$

§2 quantum argument shift algebra

~~Vinberg problem (1991)~~ Problem 1 (Vinberg, 1991)

(?) $\exists\, C_\xi \subset U\mathfrak{g}_d$ commutative subalg

s.t. $\mathrm{gr}\, C_\xi = \overline{C_\xi}$

$\longrightarrow$ Yes such C_ξ is unique

(Nazarov-Olshanski, Tarasov, Rybnikov, ...)

Def 2 C_ξ is called the quantum argument shift alg.

Problem 2

Is it also possible to lift $\overline{\partial_3}$
$$\begin{array}{c} \overline{\partial_3} \\ \cap \\ \hom U\mathfrak{g}_d \end{array}$$

$\rightarrow$ Also yes

Def 3 (Gurevich, Pyatov, and Saponov, 2012)

The quantum derivations ∂_j^i are the matrix elements of the unique linear mapping ∂

$$\begin{array}{ccccc} U\mathfrak{g}_d & \longrightarrow & M(d, U\mathfrak{g}_d) & \cong & U\mathfrak{g}_d \otimes M(d, \mathbb{C}) \\ \Downarrow & & \Downarrow & & \Downarrow \\ x & & \partial x & \cong & \sum_{i,j=1}^{d} (\partial_j^i x) \otimes E_i^j \end{array}$$

s.t.

(1) $\forall v \in \mathbb{C}$, $\partial v = 0$

(2) $\partial^i_j e^k_\ell = \delta^i_\ell \delta^k_j$

(3) (quantum Leibniz rule)

$\forall x, \forall y \in U\mathfrak{gl}_d$,

$$\partial(xy) = (\partial x)y + x(\partial y) \boxed{+ (\partial x)(\partial y)}$$

matrix product

<u>Thm 2</u> (I - Sharygin)

$$C_\xi = C[\partial_\xi x, \partial_\xi^2 x, \cdots : x \in C]$$

<u>Def 4</u>

$$C_\xi^{(n)} = C[\partial_\xi x, \cdots, \partial_\xi^n x : x \in C]$$

<u>Thm 2</u> $\Rightarrow$ $C_\xi^{(n)} \nearrow \bigcup_n C_\xi^{(n)} = C_\xi$

§3 Generators of QAS-alg.

Notation

$x \in M(d, A)$, A : alg.
$d \times d$ matrix

$$x = \begin{pmatrix} x^1_1 & \cdots & x^1_d \\ \vdots & & \vdots \\ x^d_1 & \cdots & x^d_d \end{pmatrix} = \sum_{i,j=1}^{d} x^i_j \underbrace{E^j_i}_{\text{matrix units}}$$

$i, j = 1, \dots, d$

$x^i = (x^i_1 \cdots x^i_d)$ i-th row vector

$$x_j = \begin{pmatrix} x^1_j \\ \vdots \\ x^d_j \end{pmatrix}$$ j-th column vector

Thm 3 (I)

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left(f_+^{(n-m-1)}(e)_j (e^m)^i + f_-^{(n-m-1)}(e)(e^m)^i_j \right),$$

where

$$f_\pm^{(n)}(x) \underset{\text{def}}{=} \frac{(x+1)^n \pm (x-1)^n}{2} \in \mathbb{Z}[x]$$

$$\begin{array}{cccc} \mathbb{1} : & U\mathfrak{gl} & \longrightarrow & M(d, U\mathfrak{gl}) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & x & \longmapsto & x\underline{\mathbb{1}} \end{array}$$

identity matrix

quantum Leipniz rule

$$\Rightarrow (\mathbb{1} + \partial)(xy) = (\mathbb{1} + \partial)(x)(\mathbb{1} + \partial)(y) \quad \text{---} \quad (*)$$

Convention

$$f_-^{(-1)}(x) = 1$$

$$\mathrm{tr}\, e^{-1} = 1$$

Prop 1 $\quad x = \mathrm{tr}\, e^n$

(1) $\displaystyle (1+\partial)(x) = \sum_{m=-1}^{n} \mathrm{tr}\, e^m f_-^{(n-m-1)}(e)$

(2) Mod $C_\xi^{(1)}$ $\Rightarrow C_\xi^{(1)} = \mathbb{C}[\mathrm{tr}(\xi e^n) : n = 1, 2, \dots]$

$$\partial_\xi^2 x = \sum_{m=-1}^{n} \mathrm{tr}\, e^m$$

$e^m \quad e^n$

$e^n \quad e^m$

$$\sum_{k=-1}^{n-m-1} \mathrm{tr}\left(\xi\left(\partial\, \mathrm{tr}\left(\xi f_-^{(n-m-k-2)}(e)\right)\right) f_-^{(k)}(e)\right)$$

$$\Rightarrow C_\xi^{(2)} = C_\xi^{(1)}\Big[\mathrm{tr}(\xi(\partial\,\mathrm{tr}(\xi e^n))\, e^m) + \mathrm{tr}(\xi(\partial\,\mathrm{tr}(\xi e^m))\, e^n) : m, n\Big]$$

Remark

Thanks to $(*)$ we can generalize Prop 1 appropriately for $\displaystyle x = \prod_{k=1}^{m} \mathrm{tr}\, e^{n_k}$

$$\mathrm{tr}\left(\xi\left(\partial\,\mathrm{tr}(\xi e^{n})\right)e^{m}\right)+\mathrm{tr}\left(\xi\left(\partial\,\mathrm{tr}(\xi e^{m})\right)e^{n}\right)$$

$$=\tau_\xi\begin{pmatrix}0 & P_n\\ P_m^T & 0\end{pmatrix}$$

$m\times(m+n)$ matrix

Def 5

$$P_n^{(m)}=(\underbrace{0\cdots0}_{m}\ P_n)$$

(1) $$\begin{pmatrix}f_+^{(n-1)}(x)\\ f_+^{(n-2)}(x)\\ \vdots\\ f_+^{(0)}(x)\end{pmatrix}=P_n\begin{pmatrix}1\\ x\\ \vdots\\ x^{n-1}\end{pmatrix},$$

(2) $A\in M(n,\mathbb{C})$

$$\tau_\xi(A)=\sum_{i,j=1}^{n}A^i_j\,\mathrm{tr}(\xi e^{i-1}\xi e^{j-1})$$

(3)

$$\sigma(A)=\begin{pmatrix}A^1_1 & A^1_2+A^2_1 & \cdots & A^1_n+A^n_1\\ 0 & A^2_2 & & A^2_n+A^n_2\\ \vdots & & \ddots & \\ 0 & 0 & \cdots & A^n_n\end{pmatrix}$$

Remark

$$\mathrm{tr}(\xi e^i \xi e^j) = \mathrm{tr}(\xi e^j \xi e^i), \quad \forall i, \forall j$$

$$\Rightarrow \tau_\xi(\sigma(A)) = \tau_\xi(A), \quad \forall A \in M(n, \mathbb{C})$$

$$\Rightarrow C^{(2)}_\xi = C^{(1)}_\xi \left[\tau_\xi \begin{pmatrix} 0 & P_n \\ P_m^T & 0 \end{pmatrix} : m, n = 0, 1, 2 \right]$$

$$\| $$

$$\mathrm{tr}(\xi (\partial \mathrm{tr}(\xi e^n)) e^m)$$

$$+ \mathrm{tr}(\xi (\partial \mathrm{tr}(\xi e^m)) e^n)$$

Thm 4 (I) $\forall m, \forall n$

$$\sigma\begin{pmatrix} 0 & P_{m+2n} \\ P_m^T & 0 \end{pmatrix} = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right) \times \underbrace{\boxed{P^{(m+k)}_{m+k}}}_{=\frac{1}{2}\sigma\begin{pmatrix} 0 & P_{m+k} \\ P_{m+k}^T & 0 \end{pmatrix}}$$

$$\sigma\begin{pmatrix} 0 & P_{m+2n+1} \\ P_m^T & 0 \end{pmatrix} = \sum_{k=0}^{n}\binom{2n-k}{k} \underbrace{\boxed{\left(P^{(m+k)}_{m+k+1} + P^{(m+k+1)}_{m+k}\right)}}_{=\sigma\begin{pmatrix} 0 & P_{m+k+1} \\ P_{m+k}^T & 0 \end{pmatrix}}$$

Cov

$$C_\xi^{(2)} = C_\xi^{(1)}\left[\tau_\xi(P_n^{(n)}),\ \tau_\xi(P_{n+1}^{(n)} + P_n^{(n+1)}) : n = 1, 2, \cdots\right]$$

$$\tau_\xi(P_n^{(n)}) = \frac{1}{2}\tau_\xi\begin{pmatrix} 0 & P_n \\ P_n^T & 0 \end{pmatrix}, \quad \tau_\xi(P_{n+1}^{(n)} + P_n^{(n+1)}) = \tau_\xi\begin{pmatrix} 0 & P_{n+1} \\ P_n^T & 0 \end{pmatrix}$$