§ Introduction

<u>Notation 1</u>:

$A \in M(d, U(\mathfrak{gl}(d, \mathbb{C})))$

universal enveloping alg

$\to A^i_j$: (i, j)-element of the matrix A

i.e.

$$A = \begin{pmatrix} A^1_1 & & A^1_d \\ & & \\ A^d_1 & & A^d_d \end{pmatrix}$$

<u>Def 2</u> (generating matrix)

$e \in M(d, U(\mathfrak{gl}(d, \mathbb{C})))$

$$e^i_j \overset{\text{def}}{=} \begin{array}{c c} & i \\ j & \begin{pmatrix} 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array} \quad \underline{(j, i)\text{-matrix unit}}$$

$\uparrow$ (i, j)-element of e

Def 3 (quasi-differential operators on $U(\mathfrak{gl}(d,\mathbb{C}))$)

Linear mappings

$$\partial^i_j : U(\mathfrak{gl}(d,\mathbb{C})) \to U(\mathfrak{gl}(d,\mathbb{C}))$$
$$i,j = 1,\dots,d$$

1. $\partial^i_j 1 = 0$
2. $\partial^i_j e^\ell_k = \delta^i_k \delta^\ell_j$
3. (modified Leibniz rule)

$$\partial^i_j(fg) = (\partial^i_j f)g + f(\partial^i_j g) + \sum_{k=1}^d (\partial^i_k f)(\partial^k_j g),$$
$$\forall f, \forall g \in U(\mathfrak{gl}(d,\mathbb{C}))$$

unique and well-defined.

Def 4 (argument shift)

$$\xi \in M(d,\mathbb{C})$$

$$\partial_\xi \underset{\text{def}}{=} \sum_{i,j=1}^d \xi^j_i \partial^i_j : U(\mathfrak{gl}(d,\mathbb{C})) \to U(\mathfrak{gl}(d,\mathbb{C}))$$

Conjecture 5

$$[\partial_\xi^n f, \ \partial_\xi^n g] = 0,$$

$$\forall \xi \in M(d, \mathbb{C}), \quad \forall m, \forall n \in \mathbb{Z}_+,$$

$$\forall f, \forall g \in Z(U(\mathfrak{gl}(d, \mathbb{C})))$$

§ Known results

$Z(U(\mathfrak{gl}(d,\mathbb{C})))$ is generated by

center

$\operatorname{tr} e, \ldots, \operatorname{tr} e^d$.

Thm 6

$$\partial^i_j (e^n)^k_\ell = \sum_{m=0}^{n-1} \left((e^m)^i_\ell \, f_+^{(n-m-1)}(e)^k_\ell + (e^m)^k_\ell \, f_-^{(n-m-1)}(e)^i_j \right),$$

where

$$f^{(n)}_\pm(x) = \frac{(x+1)^n \pm (x-1)^n}{2} \in \mathbb{Z}[x]$$

$$\downarrow$$

$$\xi \in U(d, \mathbb{C})$$

<u>Thm 7</u> Conjecture 5 is true for $m = n = 1$:

$$[\partial_\xi f, \partial_\xi g] = 0,$$

$$\forall f, \forall g \in Z(U(\mathfrak{gl}(d, \mathbb{C})))$$

§ (Main)

Thm 8 We have

$$[\mathrm{tr}(\xi e^m), \partial_\xi^2 \underbrace{\mathrm{tr}\, e^n}_{\text{central}}] = 0$$

for any $n \leq 7$

$$\downarrow$$

Thm 9 We have

$$[\partial_\xi f, \partial_\xi^2 \mathrm{tr}\, e^n] = 0$$

for any central element f.

for any $n \leq 7$.

$$\left(\partial_\xi f \in \mathrm{span}_{Z(U)} \{\mathrm{tr}(\xi e^m)\}_{m=0}^{\infty} \right)$$

Def 10

$$t_n \underset{\text{def}}{=} \begin{cases} 1 \\ \operatorname{tr} e^n \end{cases}$$

Using Thm 6 we obtain:

Thm 11

$$\partial_\xi^2 \operatorname{tr} e^n = \sum_{m=0}^{n-1} \Big[\partial_\xi \operatorname{tr}\big(\xi f_-^{(n-m)}(e)\big) + \sum_{(i,j)=n-m-1} \Big\{ \operatorname{tr}\big(\xi f_-^{(i)}(e)\big) \operatorname{tr}\big(\xi f_-^{(j)}(e)\big) + \operatorname{tr}\big(\xi\, \partial \operatorname{tr}\big(\xi f_-^{(i)}(e)\big) f_-^{(j)}(e)\big) \Big\} \Big] t_{m-1}$$

$$\Big\downarrow$$

Cor 12 We have $[\operatorname{tr}(\xi e^n), \partial_\xi^2 \operatorname{tr} e^n]$ for any $n \leq 7$ iff we have

$$\Big[\operatorname{tr}(\xi e^n), \overbrace{\boxed{\partial_\xi \operatorname{tr}\big(\xi f_-^{(n)}(e)\big)}}^{A_n} + \underbrace{\boxed{\sum_{k=0}^{n-1} \operatorname{tr}\big(\xi\, \partial \operatorname{tr}\big(\xi f_-^{(k)}(e)\big) f_-^{(n-k-1)}(e)\big)}}_{B_n}\Big] = 0$$

Thm 13 We have

$$[\operatorname{tr}(\xi e^m), A_n] = [\operatorname{tr}(\xi e^n), B_n] = 0$$

for any $n \leq 7$.

To prove (compute concretely) <u>Thm 8</u> the following observation is crucial.

<u>Thm 14</u> Suppose that $\xi = (\xi_1, \cdots, \xi_n)$ is a finite sequence of the set
$$M(d, c) \sqcup \{e\}.$$
We have the following:

1. $$[\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi_1 \cdots \xi_n)] = \sum_{\xi_k = e} \mathrm{tr}(\xi_1 \cdots \xi_{k-1} [e^m, \xi] \xi_{k+1} \cdots \xi_n)$$

2. $$\mathrm{tr}[\xi e^m, \xi_1 \cdots \xi_n] = \sum_{\xi_k = e} \big(\mathrm{tr}(\xi_1 \cdots \xi_{k-1} e^m) \mathrm{tr}(\xi \xi_{k+1} \cdots \xi_n) - \mathrm{tr}(\xi_1 \cdots \xi_{k-1}) \mathrm{tr}(\xi e^m \xi_{k+1} \cdots \xi_n) \big)$$

Proof. Using the commutation relation
$$[e^i_j, e^k_l] = \delta^i_l e^k_j - e^i_l \delta^k_j.$$
$\boxed{}$

Prop 15 We have

$$\partial_\xi \operatorname{tr}(\xi f_-^{(n)}(e)) = \sum_{i+j=0}^{n-1} \left\{ \begin{matrix} n \\ i \quad j \end{matrix} \right\} \operatorname{tr}(\xi e^i \xi e^j)$$

modulo $\operatorname{span}_{\mathbb{Z}} \{ \underbrace{\operatorname{tr}(\xi e^i) \operatorname{tr}(\xi e^j)}_{\text{commutes with } \operatorname{tr}(\xi e^n)} \}_{i,j=0}^{\infty}$,

where

$$\left\{ \begin{matrix} n \\ i \quad j \end{matrix} \right\} = \frac{1+(-1)^{n-i-j}}{2} \sum_{k=0}^{n-i-j-1} \underbrace{\boxed{\frac{1+(-1)^k}{2}}}_{k:\text{ even}}$$

$$\binom{n}{i+j+k+1} \binom{j+k}{j}$$

Prop 16 We have

$$\sum_{i+j=2} \begin{Bmatrix} 6 \\ i\ j \end{Bmatrix} [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^i \xi e^j)]$$

$$= 6\, [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^2)]$$

$$\sum_{i+j=4} \begin{Bmatrix} 6 \\ i\ j \end{Bmatrix} [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^i \xi e^j)]$$

$$= -6\, [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^2)]$$

$$\sum_{i+j=3} \begin{Bmatrix} 7 \\ i\ j \end{Bmatrix} [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^i \xi e^j)]$$

$$= 14\, [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^3)]$$

$$\sum_{i+j=5} \begin{Bmatrix} 7 \\ i\ j \end{Bmatrix} [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^i \xi e^j)]$$

$$= -14\, [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^3)]$$

Other terms are trivial $(=0)$

up to $n = 7$.

$$\sum_{i+j=3} \left\{ {7 \atop i\ \ j} \right\} [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^i \xi e^j)] = (*)$$

$$= 14\, [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^3)] \qquad (*)$$

$$\left\{ {7 \atop 3\ \ 0} \right\} = 42, \quad \left\{ {7 \atop 2\ \ 1} \right\} = 56,$$

$$\left\{ {7 \atop 1\ \ 2} \right\} = 77, \quad \left\{ {7 \atop 0\ \ 3} \right\} = 105$$

1

$$\sum_{i+j=3} \boxed{\left\{ {5 \atop i\ \ j} \right\}} [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^i \xi e^j)] = 0$$

$$\begin{array}{c} \| \\ 5 \end{array}$$

2

Thm 14 $\Rightarrow$ $\mathrm{tr}(\xi e^i \xi e^j)$

$$= \mathrm{tr}(\xi e^j \xi e^i)$$

$$\Rightarrow (*) = 147\, [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^3)]$$

$$+ 133\, [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^2 \xi e)]$$

(from 2)

$$= 14\, [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^3)]$$

(from 1)

Proof of 2

$$\sum_{i+j=3} [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^i \xi e^j)] = 0$$

$$\parallel$$

$$2[\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^3) + \mathrm{tr}(\xi e^2 \xi e)]$$

Thm 14

$$\Rightarrow [\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi^2 e^3)]$$

$$= \mathrm{tr}(\xi^2 [e^m, \xi] e^2) + \mathrm{tr}(\xi^2 e [e^m, \xi] e) + \mathrm{tr}(\xi^2 e^2 [e^m, \xi])$$

$$= \mathrm{tr}(\xi^2 e^m \xi e^2) + \mathrm{tr}(\xi^2 e^{m+1} \xi e) \cancel{+ \mathrm{tr}(\xi^3 e^{m+2})} \cancel{- \mathrm{tr}(\xi^3 e^{m+2})} - \mathrm{tr}(\xi^2 e \xi e^{m+1}) - \mathrm{tr}(\xi^2 e^2 \xi e^m)$$

$$[\mathrm{tr}(\xi e^m), \mathrm{tr}(\xi e^2 \xi e)]$$

$$= \mathrm{tr}(\xi [e^m, \xi] e \xi e)$$

$$+\operatorname{tr}(\xi e[e^m,\xi]\,\xi e)$$
$$+\operatorname{tr}(\xi e^2\xi\,[e^m,\xi])$$
$$=\operatorname{tr}(\xi e^m\xi e\,\xi e)$$
$$-\operatorname{tr}(\xi^2e^{m+1}\xi e)$$
$$+\operatorname{tr}(\xi e^{m+1}\xi^2e)$$
$$-\operatorname{tr}(\xi e\,\xi e^m\xi e)$$
$$+\operatorname{tr}(\xi e^2\xi e^m)$$
$$-\operatorname{tr}(\xi e^2\xi^2e^m)$$

$$\Rightarrow [\operatorname{tr}(\xi e^m),\ \operatorname{tr}(\xi^2e^3+\xi e^2\xi e)]$$
$$=\operatorname{tr}[\xi^2e^m,\ \xi e^2]$$
$$\cancel{+\operatorname{tr}[\xi^2e^{m+1},\ \xi e]}$$
$$+\operatorname{tr}[\xi e^{m+1},\ \xi^2e]$$
$$\cancel{+\operatorname{tr}[\xi^2e^2\ \xi]}$$
$$+\operatorname{tr}[\xi e^m\xi e,\ \xi e]$$

Thm 14.

$$\Rightarrow \operatorname{tr}[\xi^2 e^m, \xi e^2]$$

$$(\xi, e)$$

$$= \operatorname{tr}(\xi e^m)\operatorname{tr}(\xi^2 e) - (\operatorname{tr}\xi)\operatorname{tr}(\xi^2 e^{m+1})$$

$$(\xi e, 1)$$

$$\operatorname{tr}(\xi e^{m+1})(\operatorname{tr}\xi^2) - \operatorname{tr}(\xi e)\operatorname{tr}(\xi^2 e^m)$$

$$\operatorname{tr}[\xi e^{m+1}, \xi^2 e]$$

$$= \operatorname{tr}(\xi^2 e^{m+1})(\operatorname{tr}\xi) - (\operatorname{tr}\xi^2)\operatorname{tr}(\xi e^{m+1})$$

$$\operatorname{tr}[\xi e^m \xi e, \xi e] = -\operatorname{tr}[\xi e, \xi e^m \xi e]$$

$$(\xi e^{k-1}, e^{m-k}\xi e)$$

$$= -\sum_{k=1}^{m} \Big(\operatorname{tr}(\xi e^k)\operatorname{tr}(\xi e^{m-k}\xi e) - \operatorname{tr}(\xi e^{k-1})\operatorname{tr}(\xi e^{m-k+1}\xi e) \Big)$$

$$(\xi e^m \xi, 1)$$

$$- \Big(\operatorname{tr}(\xi e^m \xi e)(\operatorname{tr}\xi) - \operatorname{tr}(\xi^2 e^m)\operatorname{tr}(\xi e) \Big)$$

$$= - \operatorname{tr}(\xi e^m)\operatorname{tr}(\xi^2 e) \checkmark$$

$$\cancel{+ (\operatorname{tr}\xi)\operatorname{tr}(\xi e^m \xi e)}$$

$$\cancel{- \operatorname{tr}(\xi e^m \xi e)(\operatorname{tr}\xi)} + \operatorname{tr}(\xi^2 e^m)\operatorname{tr}(\xi e)$$

$$\operatorname{tr}[\xi^2 e^m, \xi e^2] + \operatorname{tr}[\xi e^{m+1}, \xi^2 e]$$

$$+ \operatorname{tr}[\xi e^m \xi e, \xi e]$$

$$= [\operatorname{tr}(\xi^2 e^m), \operatorname{tr}(\xi e)]$$

$$= \operatorname{tr}(\xi[e^m, \xi^2])$$

$$= \underbrace{\operatorname{tr}(\xi e^m \xi^2)}_{\substack{\shortparallel \\ \operatorname{tr}(\xi^3 e^m)}} - \operatorname{tr}(\xi^3 e^m)$$

$$= 0 \qquad \blacksquare$$