

Example

$$\begin{aligned}
 \int \text{tr } e^{\beta} &= \sum_{p, q} \int_p^q d\bar{q} \sum_{\bar{i}, \bar{j}, \bar{k}} e_{\bar{i}}^{\bar{j}} e_{\bar{k}}^{\bar{j}} e_{\bar{i}}^{\bar{k}} \\
 &= \sum_{p, q, \bar{i}, \bar{j}, \bar{k}} \int_p^q \left((\partial_{\bar{q}}^p e_{\bar{j}}^{\bar{i}}) e_{\bar{k}}^{\bar{j}} e_{\bar{i}}^{\bar{k}} \right. \\
 &\quad \left. + e_{\bar{j}}^{\bar{i}} (\partial_{\bar{q}}^p e_{\bar{k}}^{\bar{j}}) e_{\bar{i}}^{\bar{k}} \right. \\
 &\quad \left. + e_{\bar{j}}^{\bar{i}} e_{\bar{k}}^{\bar{j}} (\partial_{\bar{q}}^p e_{\bar{i}}^{\bar{k}}) \right) \\
 &+ \sum_{p, q, r, \bar{i}, \bar{j}, \bar{k}} \int_p^q \left((\partial_{\bar{r}}^p e_{\bar{j}}^{\bar{i}}) (\partial_{\bar{q}}^r e_{\bar{k}}^{\bar{j}}) e_{\bar{i}}^{\bar{k}} \right. \\
 &\quad \left. + (\partial_{\bar{r}}^p e_{\bar{j}}^{\bar{i}}) e_{\bar{k}}^{\bar{j}} (\partial_{\bar{q}}^r e_{\bar{i}}^{\bar{k}}) \right. \\
 &\quad \left. + e_{\bar{j}}^{\bar{i}} (\partial_{\bar{r}}^p e_{\bar{k}}^{\bar{j}}) (\partial_{\bar{q}}^r e_{\bar{i}}^{\bar{k}}) \right) \\
 &+ \sum_{p, q, r, s, \bar{i}, \bar{j}, \bar{k}} \int_p^q (\partial_{\bar{r}}^p e_{\bar{j}}^{\bar{i}}) (\partial_{\bar{s}}^r e_{\bar{k}}^{\bar{j}}) (\partial_{\bar{q}}^s e_{\bar{i}}^{\bar{k}}) \\
 &= \text{tr}(\beta e^2) + \text{tr}(e \beta e) + \text{tr}(e^2 \beta) \\
 &+ (\text{tr } \beta) (\text{tr } e) + d \text{tr}(\beta e) + (\text{tr } \beta) (\text{tr } e) \\
 &+ \text{const} \\
 \text{tr}(e \beta e) &= \sum_{\bar{i}, \bar{j}, \bar{k}} e_{\bar{j}}^{\bar{i}} \int_{\bar{k}}^{\bar{j}} e_{\bar{i}}^{\bar{k}} \quad e_{\bar{j}}^{\bar{k}} - e_{\bar{i}}^{\bar{j}} \delta_{\bar{j}}^{\bar{k}} \\
 &= \sum_{\bar{i}, \bar{j}, \bar{k}} \int_{\bar{k}}^{\bar{j}} \left(e_{\bar{i}}^{\bar{k}} e_{\bar{j}}^{\bar{i}} + [e_{\bar{j}}^{\bar{i}}, e_{\bar{i}}^{\bar{k}}] \right)
 \end{aligned}$$

$$= \text{tr}(\xi e^2) + d \text{tr}(\xi e) - (\text{tr} \xi)(\text{tr} e)$$

$$\Rightarrow d_3 \text{tr} e^3 = 3 \text{tr}(\xi e^2) + (\text{tr} \xi)(\text{tr} e) + 2d \text{tr}(\xi e)$$

$$\frac{d_3^2 \text{tr} e^4}{2} = 4 \text{tr}(\xi^2 e^2) + 2 \text{tr}(\xi e)^2 + 5d \text{tr}(\xi^2 e) + 4(\text{tr} \xi) \text{tr}(\xi e)$$

$$d_3^2 (\text{tr} L^2)^2 = 8 (\text{tr}(\xi e))^2 + 8d (\text{tr} \xi) \text{tr}(\xi e) + 16 \text{tr}(\xi^2 e)$$

$$\Rightarrow [d_3^2 \text{tr} e^4, d_3^2 (\text{tr} e^2)^2] = 0$$

Thm $[d_3^m \text{tr} e^p, d_3^n \text{tr} e^q] = 0$
for $\forall m, \forall n$ and $\forall p, \forall q \leq 4$

Conjecture

$$[d_3^m f, d_3^n g] = 0, \\ \forall m, \forall n \text{ and } \forall f, \forall g \in \mathcal{Z}(U)$$