§ <u>Notation</u> : $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$\begin{array}{c} M(d,\mathbb{F}) \\ \downarrow \\ U(M(d,\mathbb{F})) \end{array} \quad \text{canonical injective hom of Lie algebras}$$

matrix unit

$$e_i^j = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} & \vdots & \\ \cdots & 1 & \\ & & \end{pmatrix} \end{array}$$

$$[e_p^q, e_r^s] = \delta_r^q e_p^s - e_r^q \delta_p^s$$

$$e = \begin{pmatrix} e_1^1 & \cdots & e_d^1 \\ \vdots & & \vdots \\ e_1^d & \cdots & e_d^d \end{pmatrix} \in M(d, U(M(d,\mathbb{F})))$$

$$= \sum_{i,j=1}^{d} e_j^i \otimes e_i^j \in U(M(d,\mathbb{F})) \otimes M(d,\mathbb{F})$$

$$\begin{array}{cc} \cap & \cap \\ U(M(d,\mathbb{F})) & M(d,\mathbb{F}) \end{array}$$

$$(e^2)_j^i = \sum_{k=1}^{d} e_k^i e_j^k \quad \text{etc}$$

$$[e_p^q, (e^n)_r^s] = [(e^n)_p^q, e_r^s] = \delta_r^q (e^n)_p^s - (e^n)_r^q \delta_p^s$$

(quasi) differentiation

$$\partial_j^i(fg) = (\partial_j^i f)g + f(\partial_j^i g) + \sum_{k=1}^{d} (\partial_k^i f)(\partial_j^k g)$$

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§ Introduction

To compute $\partial^i_j f$ for $\forall f \in Z(U(M(d,\mathbb{F})))$,

it is desirable that we have $\partial_j^i \operatorname{tr} e^n$

in the form of :

$$\partial_j^i \operatorname{tr} e^n = A(n)_j^i, \quad \text{where}$$

$$A(n) = \sum_k \underbrace{f_{n,k}}_{\in\, Z(U(M(d,\mathbb{F})))} e^k$$

since then

$$\partial_j^i \left((\operatorname{tr} e^m)(\operatorname{tr} e^n) \right)$$

$$= \left(\partial_j^i \operatorname{tr} e^m \right) \left(\operatorname{tr} e^n \right)$$

$$+ \left(\operatorname{tr} e^m \right) \left(\partial_j^i \operatorname{tr} e^n \right)$$

$$+ \sum_{k=1}^{d} \left(\partial_k^i \operatorname{tr} e^m \right) \left(\partial_j^k \operatorname{tr} e^n \right)$$

$$= A(m)_j^i \, (\operatorname{tr} e^n) + (\operatorname{tr} e^m) \, A(n)_j^i$$

$$+ \sum_{k=1}^{d} A(m)_k^i \, A(n)_j^k$$

$$= \left((\operatorname{tr} e^n) A(m) + (\operatorname{tr} e^m) A(n) + A(m) A(n) \right)_j^i$$

$$= \Big(\sum_k (\operatorname{tr} e^n) f_{m,k} \, e^k + \sum_k (\operatorname{tr} e^m) f_{n,k} \, e^k$$

$$+ \sum_{k,l} f_{m,k} \, f_{n,k} \, e^{k+l} \Big)_j^i$$

§ Recursive formula for $\partial^i_j (e^n)^p_q$

on $n = 0, 1, 2, \ldots$

$n = 0$

$$e^0 = \delta \text{ (identity matrix)}$$

$$\partial^i_j (e^0)^p_q = \partial^i_j \delta^p_q = 0.$$

$n > 0$

$$\partial^i_j (e^n)^p_q = \sum_{v=1}^{d} \partial^i_j \left(e^p_v (e^{n-1})^v_q\right)$$

$$= \sum_{v=1}^{d} \underbrace{\left(\partial^i_j e^p_v\right)}_{\delta^i_v \delta^p_j} (e^{n-1})^v_q$$

$$+ \sum_{v=1}^{d} e^p_v \, \partial^i_j (e^{n-1})^v_q$$

$$+ \sum_{k=1}^{d} \sum_{v=1}^{d} \underbrace{\left(\partial^i_k e^p_v\right)}_{\delta^i_v \delta^p_k} \left(\partial^k_j (e^{n-1})^v_q\right)$$

$$= (e^{n-1})^i_q \delta^p_j + \sum_{v=1}^{d} e^p_v \, \partial^i_j (e^{n-1})^v_q$$

$$+ \partial^p_j (e^{n-1})^i_q$$

Lemma $\partial^i_j (e^n)^p_q$ has the following form:

$$\partial^i_j (e^n)^p_q = \sum_{\substack{k,l \geq 0 \\ k+l < n}} \left(a^{(n)}_{k,l} (e^k)^i_q (e^l)^p_j + b^{(n)}_{k,l} (e^k)^p_q (e^l)^i_j \right),$$

where $a^{(n)}_{k,l}$, $b^{(n)}_{k,l}$ are non-negative integers $\boxtimes$

Remark

Lemma $\Rightarrow$ $\partial^i_j \operatorname{tr} e^n$

$$= \sum_{\substack{k,l \geq 0 \\ k+l < n}} \left(a^{(n)}_{k,l} (e^{k+l})^i_j + (\operatorname{tr} e^k)\, b^{(n)}_{k,l} (e^l)^i_j \right)$$

$\boxtimes$

Proof of Lemma

Induction on n

$n=0$ $\quad \partial^i_j (e^0)^p_q = 0$ clear

Assume that $n>0$ and

$$\partial^i_j (e^{n-1})^p_q = \sum_{k,l} \Big(a^{(n-1)}_{k,l} (e^k)^i_q (e^l)^p_j + b^{(n-1)}_{k,l} (e^k)^p_q (e^l)^i_j \Big)$$

Then

$$\partial^i_j (e^n)^p_q = (e^{n-1})^i_q \delta^p_j + \sum_{r=1}^{d} e^p_r\, \partial^i_j (e^{n-1})^r_q + \partial^p_j (e^{n-1})^i_q$$

$$= (e^{n-1})^i_q \delta^p_j + \sum_{k,l} a^{(n-1)}_{k,l} \boxed{\sum_{r=1}^{d} e^p_r (e^k)^i_q (e^l)^r_j}$$

$$+ \sum_{k,l} b^{(n-1)}_{k,l} (e^{k+1})^p_q (e^l)^i_j$$

$$+ \sum_{k,l} \Big(a^{(n-1)}_{k,l} \underline{(e^k)^p_q (e^l)^i_j} + b^{(n-1)}_{k,l} (e^k)^i_q (e^l)^p_j \Big)$$

$$\sum_{\nu=1}^{d} e^{p}_{\nu} (e^{k})^{i}_{q} (e^{\ell})^{\nu}_{j}$$

$$= (e^{k})^{i}_{q} (e^{\ell+1})^{p}_{j} + \sum_{\nu=1}^{d} \underbrace{[e^{p}_{\nu}, (e^{k})^{i}_{q}]}_{\delta^{p}_{q} (e^{k})^{i}_{\nu} - (e^{k})^{p}_{q} \delta^{i}_{\nu}} (e^{\ell})^{\nu}_{j}$$

$$= (e^{k})^{i}_{q} (e^{\ell+1})^{p}_{j} + \delta^{p}_{q} (e^{k+\ell})^{i}_{j} \underline{- (e^{k})^{p}_{q} (e^{\ell})^{i}_{j}}$$

$$\delta^{i}_{j} (e^{n})^{p}_{q} = (e^{n-1})^{i}_{q} \delta^{p}_{j}$$
$$+ \sum_{k,\ell} a^{(n-1)}_{k,\ell} \left((e^{k})^{i}_{q} (e^{\ell+1})^{p}_{j} + \delta^{p}_{q} (e^{k+\ell})^{i}_{j} \right)$$
$$+ \sum_{k,\ell} b^{(n-1)}_{k,\ell} \left((e^{k+1})^{p}_{q} (e^{\ell})^{i}_{j} + (e^{k})^{i}_{q} (e^{\ell})^{p}_{j} \right)$$
$$\text{— } (*) \qquad \blacksquare$$

The formula $(*)$ gives us recursive determination

$$a^{(n-1)}_{k,\ell}, b^{(n-1)}_{k,\ell} \to a^{(n)}_{k,\ell}, b^{(n)}_{k,\ell} \to \cdots$$

§ Polynomials associated with $\partial_j \operatorname{tr} e^n$

Since

$$\partial_j \operatorname{tr} e^n = \sum_{\substack{k,\ell \geq 0 \\ k+\ell < n}} \left(a_{k,\ell}^{(n)} (e^{k+\ell})_j^i \right.$$

$$\left. + (\operatorname{tr} e^k)\, b_{k,\ell}^{(n)} (e^\ell)_j^i \right),$$

we define polynomials $f^{(n)}, g_k^{(n)} \in \mathbb{Z}[x]$ $(k < n)$ by

$$f^{(n)}(x) = \sum_{k,\ell} a_{k,\ell}^{(n)} x^{k+\ell}$$

$$g_k^{(n)}(x) = \sum_\ell b_{k,\ell}^{(n)} x^\ell.$$

Then we have

$$\partial_j \operatorname{tr} e^n = f^{(n)}(e)_j^i + \sum_{k<n} (\operatorname{tr} e^k)\, g_k^{(n)}(e)_j^i$$

The formula (*) gives us the recursion

$$f^{(n)}(x) = x^{n-1} + \sum_{k,l} a_{k,l}^{(n-1)} x^{k+l+1} + \sum_{k,l} b_{k,l}^{(n-1)} x^{k+l}$$

$$= x^{n-1} + x f^{(n-1)}(x) + \sum_{k<n-1} x^k g_k^{(n-1)}(x)$$

$$g_0^{(n)}(x) = \sum_{k,l} a_{k,l}^{(n-1)} x^{k+l} = f^{(n-1)}(x)$$

$$g_{k+1}^{(n)}(x) = \sum_{k,l} b_{k,l}^{(n-1)} x^{l} = g_k^{(n-1)}(x)$$

$(0 \leq k < n-1)$

$$= \cdots = g_0^{(n-k-1)}(x)$$

$$= f^{(n-k-2)}(x)$$

Therefore, we obtained

$$\partial^i_j \operatorname{tr} e^n = \left(f^{(n)}(e) + \sum_{k=0}^{n-1} (\operatorname{tr} e^k)\, f^{(n-k-1)}(e) \right)^i_{\ j}$$

and

$$f^{(n)}(x) = x^{n-1} + x f^{(n-1)}(x) + \sum_{k=0}^{n-2} x^k f^{(n-k-2)}(x).$$

$$f^{(0)}(x),\ f^{(1)}(x),\ \cdots$$

$$= 0,\ 1,\ 2x,\ 3x^2+1,\ 4x^3+4x,$$

$$5x^4+10x^2+1,\ \cdots$$

$$f^{(n)}(x) = \sum_{k=\text{odd}} \binom{n}{k} x^{n-k}$$

$$= \frac{(x+1)^n - (x-1)^n}{2}$$

Thm

$$\partial^i_j \operatorname{tr} e^n = \left(\frac{(e+\delta)^n - (e-\delta)^n}{2} + \sum_{k=0}^{n-1} (\operatorname{tr} e^k) \frac{(e+\delta)^{n-k-1} - (e-\delta)^{n-k-1}}{2} \right)^i_j$$

$\blacksquare$