A new method producing argument shift algebra for small dimension

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Argument shift method on a Poisson manifold

- Let M be a smooth manifold.

  $$\text{a Poisson bracket } \{\cdot\,,\,\cdot\} \text{ on } C^\infty(M)$$
  $$\leftrightarrow \text{a bivector field } \pi \text{ on } M \text{ satisfying } [\pi,\pi]=0,$$

  where

  $$\text{Jacobi identity on } \{\cdot\,,\,\cdot\} \Leftrightarrow [\pi,\pi]=0.$$

- A Poisson bracket on $C^\infty(M)$ is given by

$$\{f,g\} = \pi^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \qquad \pi = \pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}.$$

Argument shift method on a Poisson manifold

Given

- a smooth manifold M,
- a vector field ξ on M,
- a Poisson bivector field π on M satisfying

$$[\xi, [\xi, \pi]] = 0.$$

Theorem (argument shift method)

We have

$$\{\xi^m(f), \xi^n(g)\} = 0, \qquad \forall m, \forall n$$

for any Poisson central elements f and g, i.e.,

$$\{f, h\} = 0 = \{g, h\}, \qquad \forall h \in C^\infty(M).$$

Example

- Let $\mathfrak{g}$ be a Lie algebra and e_i be its linear basis.
- A Poisson bracket on $C^\infty(\mathfrak{g}^*)$

$$\{f, g\} = C_{ij}^k e_k \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_j}, \qquad [e_i, e_j] = C_{ij}^k e_k$$

uniquely extends the Lie bracket on $\mathfrak{g}$.

Theorem (A. Mishchenko and A. Fomenko, 1978)

Let $\xi = \xi_i \frac{\partial}{\partial e_i}$ be a constant vector field on $\mathfrak{g}^*$. We have

$$\{\xi^m(f), \xi^n(g)\} = 0, \qquad \forall m, \forall n$$

for any Poisson central elements f and g in

$$S(\mathfrak{g}) = \{\, f : f \text{ is a polynomial function on } \mathfrak{g}^* \,\}.$$

Vinberg's problem

- By Poincaré–Birkhoff–Witt theorem

$$\operatorname{Gr} U(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \frac{\operatorname{Im}\left(\bigoplus_{k=0}^{n} T^k(\mathfrak{g}) \to U(\mathfrak{g})\right)}{\operatorname{Im}\left(\bigoplus_{k=0}^{n-1} T^k(\mathfrak{g}) \to U(\mathfrak{g})\right)} \simeq S(\mathfrak{g})$$

  and there are linear noncanonical isomorphisms ι of $U(\mathfrak{g})$ onto $\operatorname{Gr} U(\mathfrak{g}) \simeq S(\mathfrak{g})$.
- An argument shift algebra A_ξ associated with a vector field ξ is a subalgebra of $S(\mathfrak{g})$ generated by

$$\left\{ \xi^n(f) : f \text{ is Poisson central and } n \geq 0 \right\}.$$

  This is Poisson commutative by Mishchenko and Fomenko theorem.

Vinberg's problem

$$
\begin{array}{ccccc}
U(\mathfrak{g}) & \xrightarrow[\text{lin. noncanon. iso}]{\iota} & \operatorname{Gr} U(\mathfrak{g}) & \xrightarrow[\text{PBW}]{\sim} & S(\mathfrak{g}) \\
\uparrow & & & & \uparrow \\
\hat{A}_\xi & & \xrightarrow{\qquad\sim\qquad} & & A_\xi
\end{array}
$$

Vinberg's problem

Find one of such linear noncanonical isomorphisms ι of $U(\mathfrak{g})$ onto $S(\mathfrak{g})$ that $\hat{A}_\xi = \iota^{-1}(A_\xi)$ is a commutative subalgebra of $U(\mathfrak{g})$.

Informally speaking, we should find commuting elements in $U(\mathfrak{g})$, which coincide with elements in A_ξ in top degrees.

Result by Tarasov

Let $\mathfrak{g} = \mathfrak{gl}_n$. Elements $\sigma_1, \ldots, \sigma_n$ of $S(\mathfrak{gl}_n)$ defined by

$$\det\bigl(t - (e^i_j)^n_{i,j=1}\bigr) = t^n + \sum_{k=1}^{n} \sigma_k t^{n-k}$$

are generators of the Poisson center of $S(\mathfrak{gl}_n)$, where $(e^i_j)^n_{i,j=1}$ is a $S(\mathfrak{gl}_n)$-coefficient matrix, whose (i,j)-entry is a matrix unit e^i_j. Suppose that $\xi = \xi_1 \partial^1_1 + \cdots + \xi_n \partial^n_n$ is a diagonal vector field. Tarasov showed that

$$\bigl[\mathrm{Sym}(\xi^k \sigma_p), \mathrm{Sym}(\xi^l \sigma_q)\bigr] = 0,$$

that is, he gave a solution for Vinberg's problem.

Twisted derivations on $U(\mathfrak{gl}_n)$

Let e^i_j be a standard basis on $\mathfrak{gl}_n$. Gurevich and Saponov defined twisted derivations $\hat{\partial}^j_i$ that are linear mappings on $U(\mathfrak{gl}_n) = T(\mathfrak{gl}_n)/I$ satisfying

$$\hat{\partial}^j_i 1 = 0, \qquad \hat{\partial}^j_i e^p_q = \delta^p_i \delta^j_q$$

and the twisted Leibniz rules

$$\hat{\partial}^j_i (fg) = (\hat{\partial}^j_i f)g + f(\hat{\partial}^j_i g) + (\hat{\partial}^k_i f)(\hat{\partial}^j_k g).$$

They are well-defined.

- Maps the ideal $I = \{ xy - yx - [x, y] : x, y \in \mathfrak{gl}_n \}$ into I.
- Twisted Leibniz rules satisfy associativity.

Twisted derivations on $U(\mathfrak{gl}_n)$

Proof that $\hat{\partial}_i^j$ are well-defined.

$$\begin{aligned}
&\hat{\partial}_i^j(e_q^p e_s^r - e_s^r e_q^p - [e_q^p, e_s^r]) \\
&\qquad = (\hat{\partial}_i^k e_q^p)(\hat{\partial}_k^j e_s^r) - (\hat{\partial}_i^k e_s^r)(\hat{\partial}_k^j e_q^p) - \hat{\partial}_i^j(\delta_q^r e_s^p - e_q^r \delta_s^p) \\
&\qquad\qquad = \delta_q^r \delta_i^p \delta_s^j - \delta_s^p \delta_i^r \delta_q^j - \delta_q^r(\hat{\partial}_i^j e_s^p) + (\hat{\partial}_i^j e_q^r)\delta_s^p = 0.
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}_i^j\big((fg)h\big) &= (\hat{\partial}_i^j f)gh + f(\hat{\partial}_i^j g)h + fg(\hat{\partial}_i^j h) \\
&\quad + (\hat{\partial}_i^k f)(\hat{\partial}_k^j g)h + (\hat{\partial}_i^k f)g(\hat{\partial}_k^j h) + f(\hat{\partial}_i^k g)(\hat{\partial}_k^j h) \\
&\quad + (\hat{\partial}_i^k f)(\hat{\partial}_k^l g)(\hat{\partial}_l^j h) = \hat{\partial}_i^j\big(f(gh)\big). \quad \square
\end{aligned}$$

Saponov's result

We define $\hat{\sigma}_k = \mathrm{Sym}(\sigma_k)$ and let $\partial_\xi = \xi^i_j \hat{\partial}^j_i$. Saponov shows that

$$[\partial^p_\xi \hat{\sigma}_k, \partial^q_\xi \hat{\sigma}_l] = 0, \qquad \forall p, q, k, l.$$

Warning!

It does not follow that

$$[\partial^p_\xi f, \partial^q_\xi g] = 0, \qquad \forall p, q \in \mathbb{Z}^+, \qquad \forall f, g \in Z\big(U(\mathfrak{gl}_n)\big),$$

though $\{\hat{\sigma}_k\}_{k=1}^n$ generates the center $Z\big(U(\mathfrak{gl}_n)\big)$.

Formulation

We define $L = (e^i_j)^n_{i,j=1} \in M\big(n, U(\mathfrak{gl}_n)\big)$. Let

$$\partial_\xi = \xi_i \hat{\partial}^i_i, \qquad \xi = \mathrm{diag}(\xi_1, \dots, \xi_n).$$

- $\big\{\mathrm{tr}(L^k)\big\}_{k=1}^n$ generates the center of $U(\mathfrak{gl}_n)$.
- I proved

$$[\partial^p_\xi \,\mathrm{tr}\, L^k, \partial^q_\xi \,\mathrm{tr}\, L^l] = 0, \qquad \forall p, q, k, l \leq 4.$$

Example of calculations

For example

$$[\partial_\xi \operatorname{tr} L^3, \partial_\xi \operatorname{tr} L^4] = 0.$$

We have modulo center the following.

$$\begin{aligned}
\partial_\xi \operatorname{tr} L^3 &= \xi_p \hat{\partial}^p_p (e^i_j e^j_k e^k_i) \\
&= \xi_p e^p_k e^k_p + \xi_p e^i_j \hat{\partial}^p_p (e^j_k e^k_i) + \xi_p (\hat{\partial}^q_p e^i_j)\big(\hat{\partial}^p_q (e^j_k e^k_i)\big) \\
&= \operatorname{tr}(\xi L^2) + \xi_p e^i_p e^p_i + \xi_p e^p_j e^j_p + \xi_p e^i_j (\hat{\partial}^q_p e^j_k)(\hat{\partial}^p_q e^k_i) + \xi_p \hat{\partial}^p_q (e^q_k e^k_p) \\
&= 2 \operatorname{tr}(\xi L^2) + \operatorname{tr}(L \xi L) + 2n \operatorname{tr}(\xi L).
\end{aligned}$$

Using commutation relation, we obtain

$$\partial_\xi \operatorname{tr} L^3 = 3 \operatorname{tr}(\xi L^2) + n \operatorname{tr}(\xi L).$$

Applying the twisted Leibniz rule, we have

$$\begin{aligned}
\hat{\partial}_\xi \operatorname{tr} L^4 = \xi_p \hat{\partial}^p_p \Big((L^2)^i_j (L^2)^j_i \Big) &= \xi_p \Big(\hat{\partial}^p_p (L^2)^i_j \Big) (L^2)^j_i \\
&+ \xi_p (L^2)^i_j \Big(\hat{\partial}^p_p (L^2)^j_i \Big) + \xi_p \Big(\hat{\partial}^q_p (L^2)^i_j \Big) \Big(\hat{\partial}^p_q (L^2)^j_i \Big).
\end{aligned}$$

Example of calculations

Substituting $\hat{\partial}_p^q (L^2)^i_j = \hat{\partial}_p^q (e^i_k e^k_j) = \delta^i_p e^q_j + e^i_p \delta^q_j + n\delta^i_p \delta^q_j$, we obtain

$$\hat{\partial}_\xi \operatorname{tr} L^4 = 2\operatorname{tr}(\xi L^3) + \operatorname{tr}(L\xi L^2) + \operatorname{tr}(L^2 \xi L) + 2n \operatorname{tr}(\xi L^2) \\ + 2\operatorname{tr}(\xi L)\operatorname{tr} L + n\operatorname{tr}(L\xi L) + 2n^2 \operatorname{tr}(\xi L).$$

Using commutation relation, finally we obtain

$$\hat{\partial}_\xi \operatorname{tr} L^4 = 4\operatorname{tr}(\xi L^3) + n\operatorname{tr}(\xi L^2) + 2\operatorname{tr}(\xi L)\operatorname{tr} L + n^2 \operatorname{tr}(\xi L)$$

modulo center. One can show that

$$\left[\operatorname{tr}(\xi^p L), \operatorname{tr}(\xi L^q)\right] = 0.$$

for any p and q.

Example of calculations and general conjecture

Therefore, we have

$$\frac{\left[\partial_\xi \operatorname{tr} L^3, \partial_\xi \operatorname{tr} L^4\right]}{12} = \left[\operatorname{tr}(\xi L^2), \operatorname{tr}(\xi L^3)\right] = -\sum_{i,j=1}^{n} \xi_i \xi_j \left[(L^3)^i_i, (L^2)^j_j\right]$$

$$= -\sum_{i,j=1}^{n} \xi_i \xi_j \left(-(L^3)^j_i e^i_j + e^j_i (L^3)^i_j\right) = 0.$$

General conjecture:

For any central elements f and g of $U(\mathfrak{gl}_n)$,

$$\left[\hat{\partial}^p_\xi f, \hat{\partial}^q_\xi g\right] = 0, \qquad \forall p, q.$$

Case: $\mathfrak{g} = \mathfrak{so}_{2n}$

Consider $\mathfrak{g} = \mathfrak{so}_{2n}$ and let $f^i_j = e^i_j - e^j_i$. Elements $\sigma_2, \ldots, \sigma_{2n}$ are defined by

$$\det\bigl(t - (f^i_j)^n_{i,j=1}\bigr) = t^{2n} + \sum_{k=1}^{n} \sigma_{2k} t^{n-2k}$$

and we can write $\sigma_{2n} = \mathrm{Pf}^2$. We have $\sigma_2, \ldots, \sigma_{2n-2}$, Pf as generators of the Poisson center of $S(\mathfrak{g})$. Also we define twisted derivations $\hat{\partial}^i_j$, $i < j$ on $U(\mathfrak{g})$ by imbedding $U(\mathfrak{g})$ in $U(\mathfrak{gl}_{2n})$. It is not known if

$$\bigl[\partial^p_\xi \mathrm{Sym}(f), \partial^q_\xi \mathrm{Sym}(g)\bigr] = 0, \qquad \forall p, q,$$

where $\partial_\xi = \xi^i_j \hat{\partial}^j_i$, f, $g \in \{\sigma_2, \ldots, \sigma_{2n-2}, \mathrm{Pf}\}$, is true.